Operators on Hilbert Space

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Abstract

The main object of this thesis is the spectral theorem for self-adjoint operators on Hilbert space. We begin with a thorough survey of preliminary Hilbert space theory including the Riesz representation theorem, sequential weak compactness of the closed unit ball, and many other key results, before moving on to the study of bounded linear operators between Hilbert spaces. We then look at compact, self-adjoint operators and establish a version of the spectral theorem for them, before proving the spectral theorem in more generality for self-adjoint operators.

Contents

1 Introduction

The theory of Hilbert spaces generalises the theory of Euclidean finite-dimensional spaces to infinite dimensions. The notion of distance in a Hilbert space is connected to the concept of an inner product and consequently, many geometric notions arise, allowing for a far more rigid structure than is available in a more general Banach space. Although the axioms of a Hilbert space are simple, the theory is a far-reaching subject in modern mathematics, with a large number of applications; Quantum Mechanics, where the self-adjoint operators serve as the observables, in particular has made notable use. In this thesis, we begin with the fundamental theory of Hilbert spaces, before developing an understanding of operators on Hilbert spaces, which will allow us to state and prove the spectral theorem, which is the culmination of this project.

In chapter 1, we begin by first formulating the concept of an inner product space, and then introduce a Hilbert space as an inner product space in which the induced metric is complete. Some familiar examples are shown such as $\mathbb{C}^n$ with the familiar "dot product", as well as the collection of absolutely square summable sequences ℓ^2. We also provide some non-examples, before developing the important concept of a Hilbert space basis, alternatively referred to as an orthonormal basis. This notion of basis allows us to compute elements of the Hilbert space as infinite sums of the basis elements, which turns out to be very important in later chapters.
In the final sections of the chapter, we define various types of isomorphisms between Hilbert spaces and show that every separable Hilbert space is isometrically isomorphic to ℓ^2. In the last section, we first prove the important Riesz representation theorem, from which the Hahn–Banach theorem for Hilbert spaces follows as a result. We also define weak convergence and show that the closed unit ball of a Hilbert space is sequentially weakly compact.

In the second chapter, we begin the theory of operators, which are bounded linear maps from one Hilbert space to another, although we mainly focus on operators going to and from the same Hilbert space. The Riesz representation theorem from chapter 1 provides for each operator the existence of the adjoint, a very important notion used extensively afterwards. The most important class of operators we study are the self-adjoint ones - those that are equal to their own adjoint. We also look at other interesting types of operators, including finite rank, compact, unitary, and normal operators, and study some properties. Compact operators in particular are of interest, and we characterise some equivalent notions of compact operators before ending the chapter by showing that the adjoint of a compact operator is also compact.

In the final chapter on spectral theory, we begin by reviewing the concepts of eigenvalues and eigenvectors and show a new theory is needed for infinite dimensions as not all operators have eigenvalues, in particular, the right shift is shown as an example. We therefore define the spectrum of an operator A as the set of all $\lambda \in \mathbb{C}$ such that $A - \lambda I$ is not bijective. We explore some spectrum properties and show that it is a compact subset of $\mathbb{C}$ contained in the circle of radius $\|A\|$ centred about the origin.
In the second section, we use much of the theory developed in earlier chapters to state and prove the spectral theorem for compact self-adjoint operators, which provides an analogue of the familiar spectral theorem from finite-dimensional linear algebra.
In the last section, we state and prove the spectral theorem in more generality, requiring our operator only the be self-adjoint and dropping compactness. To do this, we introduce the notion of a projection-valued measure, utilising measure and integration theory. We also prove the existence of a functional calculus for each operator and explore the associated properties. Many key concepts from measure theory are used throughout, such as monotone classes and Lebesgue's dominated convergence theorem.

The purpose of the appendix is to present key results throughout mathematics of which we will make frequent use, but whose proofs would lead us too far away from the main objective of this thesis.

2 Geometry of Hilbert Space

In this chapter, we discuss the elementary theory of Hilbert Spaces and examine the geometry behind them. Throughout, we will mostly deal with complex vector spaces unless stated otherwise. The first chapter of [6] and chapter three of [8] are used as guiding sources.

2.1 Inner Product Spaces

Definition 2.1 (Inner Product)**.** Let E be a complex vector space. An inner product on E is a function $(\cdot \mid \cdot) : E \times E \to \mathbb{C}$ that satisfies the following conditions;

(i) *Conjugate Symmetry*
$(x \mid y) = \overline{(y \mid x)}$ for all $x, y \in E$.

(ii) *Linearity in the first component*
$(\alpha x_1 + \beta x_2 \mid y) = \alpha(x_1 \mid y) + \beta(x_2 \mid y)$ for all $x_1, x_2, y \in E$ and $\alpha, \beta \in \mathbb{C}$.

(iii) *Positive Definiteness*
$(x \mid x) \geq 0$ for all $x \in E$ and $(x \mid x) = 0$ if and only if $x = 0$.

Remark 2.2. Condition (i) is also known as *Hermitian Symmetry.*

Remark 2.3. Conditions (i) and (ii) show that $(\cdot \mid \cdot)$ is antilinear with respect to the second argument,

$$\begin{aligned}(x \mid \alpha y_1 + \beta y_2) &= \overline{(\alpha y_1 + \beta y_2 \mid x)} \\ &= \overline{(\alpha y_1 \mid x) + (\beta y_2 \mid x)} \\ &= \overline{\alpha}\,\overline{(y_1 \mid x)} + \overline{\beta}\,\overline{(y_2 \mid x)} \\ &= \overline{\alpha}(x \mid y_1) + \overline{\beta}(x \mid y_2).\end{aligned}$$

This shows that an inner product is in fact an example of a sesquilinear form on E. The pair $(E, (\cdot \mid \cdot))$ will together be known as an *inner product space.*

Example 2.4. The prototypical example of an inner product space is given by the complex vector space

$$\mathbb{C}^n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{C}, i = 1, 2, \ldots, n\}$$

together with the inner product $(\cdot \mid \cdot) : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ given by

$$(x \mid y) := \sum_{i=1}^{n} x_i \overline{y_i}, \text{ for } x, y \in \mathbb{C}^n$$

To see that this is an inner product space, let $\alpha, \beta \in \mathbb{C}$, $x, y, z \in \mathbb{C}^n$ and let $x = (x_i)_{i=1}^n, y = (y_i)_{i=1}^n, z = (z_i)_{i=1}^n$. We then have the following.

(i)

$$(x \mid y) = \sum_{i=1}^{n} x_i \overline{y_i} = \overline{\sum_{i=1}^{n} y_i \overline{x_i}} = \overline{(y \mid x)}$$

(ii)

$$(\alpha x + \beta y \mid z) = \sum_{i=1}^{n} (\alpha x_i + \beta y_i) \overline{z_i} = \alpha \sum_{i=1}^{n} x_i \overline{z_i} + \beta \sum_{i=1}^{n} y_i \overline{y_i} = \alpha(x \mid z) + \beta(y \mid z)$$

(iii)

$$\sum_{i=1}^{n} x_i \overline{x_i} \geq 0 \text{ since } x_i \overline{x_i} \geq 0 \text{ for all complex numbers.}$$

$$\text{If } \sum_{i=1}^{n} x_i \overline{x_i} = 0 \text{ then } x_i = 0 \text{ for all } i \in \{1, 2, \dots, n\} \text{ and so } x = 0.$$

Thus the three conditions of an inner product are satisfied and $(\mathbb{C}^n, (\cdot \mid \cdot))$ is indeed an inner product space.

Example 2.5. The set of absolutely square summable sequences

$$\ell^2 = \{(x_n)_{n \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \mid \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$$

is a subset of $\mathbb{C}^{\mathbb{N}}$, the vector space of all sequences of complex numbers. To see that ℓ^2 is also a vector space, it suffices to show that $(0)_{n \in \mathbb{N}} \in \ell^2$, and that for any $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \in \ell^2$ and $\lambda \in \mathbb{C}$, we have both $(\lambda x_n)_{n \in \mathbb{N}}$ and $(x_n + y_n)_{n \in \mathbb{N}}$ as elements of ℓ^2.
Clearly the first condition is true. For the second, a straightforward computation shows that $\sum_{i=1}^{\infty} |\lambda x_i|^2 = |\lambda|^2 \sum_{i=1}^{\infty} |x_i|^2 < \infty$, and hence $(\lambda x_n)_{n \in \mathbb{N}} \in \ell^2$.
For the third and final condition, recall Minkowski's Inequality [8] page 6, which says that for $p \geq 1$ and any two sequences of complex numbers $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$, we have the following,

$$\left(\sum_{n=1}^{\infty}|x_n+y_n|^p\right)^{\frac{1}{p}} \leq \left(\sum_{n=1}^{\infty}|x_n|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty}y_n|^p\right)^{\frac{1}{p}} \tag{2.1}$$

Thus using $p = 2$ in the above inequality gives $(x_n + y_n)_{n\in\mathbb{N}} \in \ell^2$.

For $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \in \ell^2$ we define the inner product as

$$((x_n)_{n\in\mathbb{N}} \mid (y_n)_{n\in\mathbb{N}}) := \sum_{n=1}^{\infty} x_n\overline{y_n} \tag{2.2}$$

The proof that this is in an inner product is very similar to the $\mathbb{C}^n$ case from example 2.4.

Having seen some examples of an inner product space, we will now work towards understanding the special geometric properties all inner product spaces possess.

Lemma 2.6. Let $(E, (\cdot \mid \cdot))$ be a complex inner product space. Then for all $x, y \in E$ we have the following inequality.

$$|(x \mid y)|^2 \leq (x \mid x)(y \mid y) \tag{2.3}$$

Moreover, equality holds if and only if x and y are linearly dependent.

Proof: If $(y \mid y) = 0$ then $y = 0$ and the inequality is obviously true. So suppose $(y \mid y) \neq 0$ and let $\alpha \in \mathbb{C}$ be arbitrary. We then have

$$0 \leq (x - \alpha y \mid x - \alpha y) = (x \mid x) - \alpha(y \mid x) - \overline{\alpha}(x \mid y) + \alpha\overline{\alpha}(y \mid y).$$

Let $\alpha = \frac{(x|y)}{(y|y)}$ and we have the following,

$$0 \leq (x \mid x) - \frac{(x \mid y)}{(y \mid y)}(y \mid x) - \frac{\overline{(x \mid y)}}{(y \mid y)}(x \mid y) + \frac{(x \mid y)}{(y \mid y)}\frac{\overline{(x \mid y)}}{(y \mid y)}(y \mid y)$$

This in turn gives $0 \leq (x \mid x) - \frac{(x|y)\overline{(x|y)}}{(y|y)}$, whence the desired inequality follows.

Suppose now (2.3) holds as an equality, that is, $|(x \mid y)|^2 = (x \mid x)(y \mid y)$. Note that $|(x \mid y)|^2 = (x \mid y)\overline{(x \mid y)} = (x \mid y)(y \mid x)$, allowing us to rewrite the equality as

$$(x \mid y)(y \mid x) = (x \mid x)(y \mid y) \tag{2.4}$$

Then consider the following inner product,

$$\begin{aligned}\Big((y \mid y)x - (x \mid y)y \,\Big|\, (y \mid y)x - (x \mid y)y \Big) &= (y \mid y)^2(x \mid x) - (y \mid x)(x \mid y)(y \mid y) \\ &= (y \mid y)\Big((y \mid y)(x \mid x) - (x \mid y)(y \mid x)\Big) = 0 \text{ by (2.4)}\end{aligned}$$

Thus $(y \mid y)x - (x \mid y)y = 0$ and so x and y are linearly dependent.

For the reverse, suppose x and y are linearly dependent. Then $y = \alpha x$ for some $\alpha \in \mathbb{C}$. A straightforward computation then yields the equality,

$$|(x \mid y)|^2 = |\overline{\alpha}|^2 |(x \mid x)|^2 = \alpha\overline{\alpha}(x \mid x)^2 = (x \mid x)(\alpha x \mid \alpha x) = (x \mid x)(y \mid y).$$

■

Theorem 2.7. Let E be a complex inner product space with inner product $(\cdot \mid \cdot)$. Then for all $x \in E$, the inner product induces a norm in the following way,

$$\|x\| := \sqrt{(x \mid x)}.$$

Proof: Recall the three axioms of a norm are (i) Homogenity, (ii) Positive-Definiteness, and (iii) the Triangle Inequality.

For (i) let $\alpha \in \mathbb{C}$, and then compute the following.

$$\|\alpha x\| = \sqrt{(\alpha x \mid \alpha x)} = \sqrt{\alpha\overline{\alpha}(x \mid x)} = \sqrt{|\alpha|^2(x \mid x)} = \alpha\|x\|.$$

For (ii) we use the fact that $(x \mid x) \geq 0$ is always true to get that $\|x\| = \sqrt{(x \mid x)} \geq 0$. The case $\|x\| = \sqrt{(x \mid x)} = 0$ is true if and only if $(x \mid x) = 0$ as desired.

For the triangle inequality, let $x, y \in E$. Then using 2.6 we have the following.

$$\begin{aligned}\|x+y\|^2 = (x+y \mid x+y) &= \|x\|^2 + \|y\|^2 + (x \mid y) + \overline{(x \mid y)} \\ &= \|x\|^2 + \|y\|^2 + 2\,\mathrm{Re}[(x \mid y)] \leq \|x\|^2 + \|y\|^2 + 2|(x \mid y)| \\ &\leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| \\ &= (\|x\| + \|y\|)^2 .\end{aligned}$$

■

Having shown that the norm constructed in 2.7 actually constitutes a norm, Lemma 2.6 can be reformulated as follows.

Theorem 2.8 (Cauchy–Schwarz Inequality)**.** *Let E be a complex vector space with inner product $(\cdot \mid \cdot)$ and induced norm $\|\cdot\|$. Then we have the following inequality.*

$$|(x \mid y)| \leq \|x\| \, \|y\| \text{ for all } x, y \in E.$$

Proof: Combine 2.7 with 2.6. ■

From now on, we will always equip an inner product space with the norm introduced above.

Theorem 2.9 (Parallelogram Law)**.** *For any two elements x and y in an inner product space we have*

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$$

Proof: Consider the squared norms of $x+y$ and $x-y$,

$$\|x+y\|^2 = (x+y \mid x+y) = \|x\|^2 + \|y\|^2 + (x \mid y) + (y \mid x), \tag{2.5}$$
$$\|x-y\|^2 = (x-y \mid x-y) = \|x\|^2 + \|y\|^2 - (x \mid y) - (y \mid x).$$

Adding these two equations together gives the result immediately.

■

Remark 2.10. For any complex number $z \in \mathbb{C}$, we have that $z + \overline{z} = 2\operatorname{Re}(z)$. Thus equation (2.5) in the preceding proof may be written as

$$\|x+y\|^2 = \|x\|^2 + 2\operatorname{Re}(x \mid y) + \|y\|^2. \tag{2.6}$$

This is known as the *polar identity.*

Definition 2.11. Two vectors x and y in an inner product space are *orthogonal*, denoted by $x \perp y$, if $(x \mid y) = 0$.

If $x \perp y$ then $(y \mid x) = \overline{(x \mid y)} = 0$ and so $y \perp x$. Thus, orthogonality is a symmetric relation.

Theorem 2.12 (Pythagoras)**.** *For any two orthogonal vectors in an inner product space we have the following relation.*

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 \tag{2.7}$$

Proof: If $x \perp y$ then $(x \mid y) = (y \mid x) = 0$, and the theorem follows immediately from 2.5. ■

Remark 2.13. Using the parallelogram law it is easy to find examples of norms on a complex vector space which are not induced by an inner product. For example, let ℓ^1 be the set of all absolutely-summable complex sequences, that is,

$$\ell^1 = \{(x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \mid \sum_{n=1}^{\infty} |x_n| < \infty\} \tag{2.8}$$

Then it is easy to show that ℓ^1 is a normed vector space with norm given by

$$\|(x)_{n\in\mathbb{N}}\| = \sum_{n=1}^{\infty} |x_n|. \tag{2.9}$$

Consider the the vectors $e_1 = (1, 0, 0, \dots)$ and $e_2 = (0, 1, 0, \dots)$. Then $\|e_1 + e_2\|^2 = 2^2 = 4$ and $\|e_1 - e_2\| = 2^2 = 4$. Meanwhile $\|e_1\|^2 = 1 = \|e_2\|^2$. Since $8 \neq 4$ the parallelogram law is not satisfied and thus this norm on ℓ^1 is not induced by an inner product.

It can be shown that any norm which satisfies the parallelogram law has to come from an inner product; this is the Jordan–von Neumann theorem, see section 8 in [11].

2.2 Complete Inner Product Spaces

In this section, we define what is meant by a Hilbert space and look at some examples and some non-examples. Chapter 1 of [6] and section 3.4 in [8] are both followed closely.

Definition 2.14. A complete inner product space is known as a *Hilbert Space*, and will often be denoted by an upper case H.

Example 2.4 is a Hilbert Space. This follows from the completeness of $\mathbb{C}$, which is a consequence of the Bolzano–Weierstrass theorem, see page 51 [17].
In Example 2.5 we showed that the set of square summable sequences ℓ^2 is an inner product space, in fact, it is also a Hilbert space, see [14] theorem 9.8.
We will now discuss some inner product spaces which are not Hilbert spaces.

Example 2.15. Consider the set of continuous functions on the unit interval,

$$C([0,1]) := \{f \in \mathbb{C}^{[0,1]} \mid f \text{ is continuous}\}$$

with inner product given by $(f \mid g) := \int_0^1 f(x)\overline{g(x)}dx$, for $f, g \in C([0,1])$, see section 3, [8]. To see that the space is not complete, consider the following sequence $(f_n)_{n\in\mathbb{N}} \in C([0,1])^{\mathbb{N}}$,

$$f_n(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq \frac{1}{2} \\ 1 - 2n\left(x - \frac{1}{2}\right) & \text{if } \frac{1}{2} \leq x \leq \frac{1}{2n} + \frac{1}{2} \\ 0 & \text{if } \frac{1}{2n} + \frac{1}{2} < x \leq 1 \end{cases}$$

Each f_n is continuous. To see this, let $x_0 \in [0,1]$ and suppose $\varepsilon > 0$. If x_0 is either of the endpoints, i.e. 0 or 1, it is clear that f_n is continuous at x_0 as sufficiently small x values to the right and the left from 0 and 1 respectively make $|f(0) - f(x)| = 0 = |f(1) - f(x)|$.
If $x_0 \in (0, \frac{1}{2})$, let $\delta_1 = \min\{d(x_0, \frac{1}{2}), x_0\}$, so that if $|x - x_0| < \delta_1$, then $|f(x) - f(x_0)| < \varepsilon$. If $x_0 \in [\frac{1}{2}, \frac{1}{2n} + \frac{1}{2})$, then let $\delta_2 = \frac{\varepsilon}{2n}$. We then have that if $|x - x_0| < \delta_2$, then $|f(x) - f(x_0)| < \varepsilon$. Lastly, if $x_0 \in [\frac{1}{2n} + \frac{1}{2}, 1]$, let $\delta_3 = \min\{d(x_0, \frac{1}{2n} + \frac{1}{2}), 1 - x_0\}$, which as in the previous cases gives $|f(x) - f(x_0)| < \varepsilon$ whenever $|x - x_0| < \delta_3$. Thus each f_n is continuous.
Moreover, $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence since

$$\|f_n - f_m\| = \sqrt{\int_0^1 |f_n(x) - f_m(x)|^2 dx} \leq \left(\frac{1}{n} + \frac{1}{m}\right)^{\frac{1}{2}} \longrightarrow 0.$$

However, the sequence does not converge with respect to this inner product as in the limit it tends to the following discontinuous function, which is not an element of $C([0,1])$,

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} < x \leq 1 \end{cases}$$

Consequently, $C([0,1])$ with the inner product defined as above is not a Hilbert Space. If we restrict ourselves only to the level of normed spaces and put the supremum norm on $C([0,1])$, the space turns out to be complete, that is, a Banach space. This is a standard proof, see [8] section 1.5. This norm is not induced by an inner product however. Consider the functions $f(x) = 1 - x$ and $g(x) = x$. Then

$$2 = \|f + g\|^2 + \|f - g\|^2 \neq 2\left(\|f\|^2 + \|g\|^2\right) = 4$$

Since it doesn't satisfy the parallelogram law, the uniform norm is not induced by an inner product.

Example 2.16. Another non-example is exhibited by the following space, with the same inner product as defined in 2.5.

$$\varphi_0 = \{(x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \mid \exists n_0 \in \mathbb{N} \text{ such that } x_n = 0 \text{ for all } n \geq n_0\} \tag{2.10}$$

which is a subspace of the set of all sequences that converge to zero, that is,

$$\varphi = \{(x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \mid \lim_{n\to\infty} x_n = 0\}. \tag{2.11}$$

Consider the the sequence $(\rho_n)_{n\in\mathbb{N}} \in {\varphi_0}^{\mathbb{N}}$ given by

$$\rho_n = \left(1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right)$$

Then $\lim_{n,m\to\infty} \|\rho_n - \rho_m\| = \lim_{n,m\to\infty} \left(\sum_{k=m+1}^{n} \frac{1}{k^2}\right)^{\frac{1}{2}} = 0$, for $m < n$, and so the sequence $(\rho_n)_{n\in\mathbb{N}}$ Cauchy. However, the sequence does not converge as it tends to the sequence $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ which is an element of φ, but not φ_0. Hence φ_0 is not closed in φ and therefore is not a Hilbert space.

Remark 2.17. In all of the above we have been considering complex inner product spaces. It is also natural to consider real inner product spaces, in which case we have a real vector space E and the inner product is now a map $(\cdot \mid \cdot) : E \times E \to \mathbb{R}$ that satisfies the same axioms listed in 2.1. In the real case however, the notions of antilinearity and linearity are the same, and thus an inner product on a real vector space is a symmetric bilinear form.
From now on we will mostly only consider complex inner product spaces, due to the fact that every real Hilbert space can be complexified to a complex one, see section 1 [6].

2.3 Orthogonal and Orthonormal Systems

In this section, we discuss the fundamental geometric structure that makes Hilbert spaces more special than ordinary Banach spaces. In particular, the notion of orthonormal basis which we will study is of high importance.

Definition 2.18. Let E be an inner product space. A set F of non-zero vectors in E is called an *orthogonal system* if $x \perp y$ for any two distinct elements of F. Moreover, if $\|x\| = 1$ for all $x \in F$, we call the system *orthonormal.*

Note that if x is orthogonal to each of $x_1, x_2, \ldots, x_n$ then it is also orthogonal to any linear combination of the x_i. To see this take an arbitrary linear combination $y := \sum_{i=1}^{n} \alpha_i x_i$ and consider the following.

$$(x \mid y) = \left(x \mid \sum_{i=1}^{n} \alpha_i x_i\right) = \sum_{i=1}^{n} \overline{\alpha_i}(x \mid x_i) = 0 \tag{2.12}$$

Theorem 2.19 (Generalised Pythagorean Theorem)**.** Let E be an inner product space and suppose n is any natural number greater than or equal to 2. If $\{x_1, x_2, \ldots x_n\}$ is an orthogonal system, then

$$\left\|\sum_{k=1}^{n} x_k\right\|^2 = \sum_{k=1}^{n} \|x_k\|^2$$

Proof: If $x_1 \perp x_2$ then $\|x_1 + x_2\|^2 = \|x_1\|^2 + \|x_2\|^2$ by 2.7, and so the claim is true for $n = 2$. Suppose now that the claim holds for $n-1$, that is,

$$\left\|\sum_{k=1}^{n-1} x_k\right\|^2 = \sum_{k=1}^{n-1} \|x_k\|^2$$

Let $x = \sum_{k=1}^{n-1} x_k$ and let $y = x_n$. By 2.12, $x \perp y$, and we have

$$\left\|\sum_{k=1}^{n} x_k\right\|^2 = \|x+y\|^2 = \|x\|^2 + \|y\|^2 = \sum_{k=1}^{n-1} \|x_k\|^2 + \|x_n\|^2 = \sum_{k=1}^{n} \|x_k\|^2$$

The result follows by induction.

■

Theorem 2.20 (Bessel's Equality and Inequality)**.** *Let* $\{x_1, x_2, \ldots, x_n\}$ *be an orthonormal set of vectors in an inner product space* $(E, (\cdot \mid \cdot))$. *Then for every* $x \in E$ *we have the following,*

$$\left\|x - \sum_{k=1}^{n} (x \mid x_k) x_k\right\|^2 = \|x\|^2 - \sum_{k=1}^{n} |(x \mid x_k)|^2, \tag{2.13}$$

$$\sum_{k=1}^{n} |(x \mid x_k)|^2 \leq \|x\|^2. \tag{2.14}$$

Proof: Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be arbitrary complex numbers. By 2.19, we have

$$\left\|\sum_{k=1}^{n} \alpha_k x_k\right\|^2 = \sum_{k=1}^{n} \|\alpha_k x_k\|^2 = \sum_{k=1}^{n} |\alpha_k|^2.$$

Then consider the following,

$$\begin{aligned}
\left\|x - \sum_{k=1}^{n} \alpha_k x_k\right\|^2 &= \left(x - \sum_{k=1}^{n} \alpha_k x_k \mid x - \sum_{k=1}^{n} \alpha_k x_k\right) \\
&= \|x\|^2 - \left(x \mid \sum_{k=1}^{n} \alpha_k x_k\right) - \left(\sum_{k=1}^{n} \alpha_k x_k \mid x\right) + \sum_{k=1}^{n} |\alpha_k|^2 \|x_k\|^2 \\
&= \|x\|^2 - \sum_{k=1}^{n} \overline{\alpha_k}(x \mid x_k) - \sum_{k=1}^{n} \alpha_k \overline{(x \mid x_k)} + \sum_{k=1}^{n} \alpha_k \overline{\alpha_k} \\
&= \|x\|^2 - \sum_{k=1}^{n} |(x \mid x_k)|^2 + \sum_{k=1}^{n} |(x \mid x_k) - \alpha_k|^2.
\end{aligned}$$

In particular, for $\alpha_k := (x \mid x_k)$ we have

$$\left\| x - \sum_{k=1}^{n} (x \mid x_k) x_k \right\|^2 = \|x\|^2 - \sum_{k=1}^{n} |(x \mid x_k)|^2,$$

giving the first result. Then since the norm is alwasy non-negative we have that $0 \leq \|x\|^2 - \sum_{k=1}^{n} |(x \mid x_k)|^2$ which gives the second result. ■

Definition 2.21. Let S be a non-emtpy subset of a Hilbert space H. The set of all elements of H orthogonal to S, denoted by $S^{\perp}$, is called the *orthogonal complement* of S, i.e,

$$S^{\perp} := \{x \in H : x \perp s \text{ for all } s \in S\} \tag{2.15}$$

Theorem 2.22. *For any subset S of a Hilbert space H, the set $S^{\perp}$ is a closed subspace of H.*

Proof: First note that for $\alpha, \beta \in \mathbb{C}$ and $x, y \in S^{\perp}$ we have $(\alpha x + \beta y \mid z) = \alpha(x \mid z) + \beta(y \mid z) = 0$ for all $z \in S$. Thus, $S^{\perp}$ is a subspace of H.
To see that it is closed, let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $S^{\perp}$, and let $x \in H$ be such that $\lim_{n \to \infty} x_n = x$. From the continuity of the inner product A.30, for every $y \in S^{\perp}$ we have the following,

$$(x \mid y) = \left(\lim_{n \to \infty} x_n \mid y \right) = \lim_{n \to \infty} (x_n \mid y) = 0.$$

This shows that $x \in S^{\perp}$ and hence S is closed.

■

Definition 2.23. Suppose V is a vector space, U is a subset of V, and the points x, y are in U. *The line segment between x and y*, denoted by $[x, y]$, is given by the following set,

$$[x, y] := \{(1 - \alpha)x + \alpha y \mid \alpha \in [0, 1]\}. \tag{2.16}$$

We say that U is *convex* if the line segment between any two points in U is entirely contained in U.

Theorem 2.24 (Closest Point Property)**.** *Let S be a closed convex subset of a Hilbert space H. For every point $x \in H$, there exists a unique point $y \in S$ such that $\|x - y\| = \inf\{\|x - z\| : z \in S\}$.*

Proof: Let $(y_n)_{n \in \mathbb{N}}$ be a sequence in S such that

$$\lim_{n \to \infty} \|x - y_n\| = \inf\{\|x - z\| : z \in S\}.$$

For brevity let $d = \inf\{\|x - z\| : z \in S\}$. Since $\frac{1}{2}(y_m + y_n) \in S$, we have that

$$\|x - \frac{1}{2}(y_m + y_n)\| \geq d, \text{ for all } m, n \in \mathbb{N}.$$

By the parallelogram law 2.9, we obtain the following,

$$\begin{aligned}\|y_m - y_n\|^2 &= 4\|x - \frac{1}{2}(y_m + y_n)\|^2 + \|y_m - y_n\|^2 - 4\|x - \frac{1}{2}(y_m + y_n)\|^2 \\ &= 2\left(\|x - y_m\|^2 + \|x - y_n\|^2\right) - 4\|x - \frac{1}{2}(y_m + y_n)\|^2.\end{aligned}$$

Since $2\left(\|x - y_m\|^2 + \|x - y_n\|^2\right) \to 4d^2$ as $m, n \to \infty$, and $\|x - \frac{1}{2}(y_m + y_n)\|^2 \geq d^2$, we have $\|y_m - y_n\|^2 \to 0$ as $m, n \to \infty$. Thus $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Since H is complete and S is closed, $\lim_{n\to\infty} y_n := y$ exists and is in S. Moreover, from the continuity of the norm, we have

$$\|x - y\| = \|x - \lim_{n\to\infty} y_n\| = \lim_{n\to\infty} \|x - y_n\| = d$$

For uniqueness, suppose we have another point $y' \in S$ satisfying the claim. Then since $\frac{1}{2}(y' + y) \in S$, we have that,

$$\|y - y'\|^2 = 4d^2 - 4\|x + \frac{y - y'}{2}\|^2 \leq 0.$$

This implies $y = y'$, completing the proof. ■

Theorem 2.25. *Let H be a Hilbert space and let S be a closed subspace of H. Then every element $x \in H$ has a unique decomposition of the form $x = y + z$ where $y \in S$ and $z \in S^\perp$.*

Proof: If $x \in S$ then the decomposition is clearly $x = x + 0$. Suppose therefore that $x \notin S$. Let y be the unique point of S satisfying the closest point property from theorem 2.24, i.e, y satisfies $\|x - y\| = \inf_{w\in S} \|x - w\|$. We will show that $x = y + (x - y)$ is the required decomposition.
If $w \in S$ and $\lambda \in \mathbb{C}$, then $y + \lambda w \in S$ and

$$\|x - y\|^2 \leq \|x - y - \lambda w\|^2 = \|x - y\|^2 - 2\operatorname{Re}\lambda(w \mid x - y) + |\lambda|^2\|w\|^2.$$

We therefore have that

$$-2\operatorname{Re}\lambda(w \mid x - y) + |\lambda|^2\|w\|^2 \geq 0.$$

If $\lambda > 0$, then diving by λ and letting $\lambda \to 0$ gives

$$\mathrm{Re}(w \mid x - y) \leq 0. \tag{2.17}$$

Similarly, replacing λ by $-i\lambda\,(\lambda > 0)$, dividing by λ, and letting $\lambda \to 0$ gives

$$\mathrm{Im}(w \mid x - y) \leq 0. \tag{2.18}$$

Since $y \in S$ implies $-y \in S$, inequalities (2.17) and (2.18) hold also with $-w$ instead of w. Therefore $(w \mid x - y) = 0$ for every $w \in S$, which means that $x - y \in S^{\perp}$. Thus, the decomposition exists.
For uniqueness, suppose we we have two decompositions for x, $x = y+z$ and $x = y'+z'$ where $y, y' \in S$ and $z, z' \in S^{\perp}$. We then have $y - y' \in S$ and $z' - z \in S^{\perp}$, and $y - y' = z' - z$. Since $S \cap S^{\perp} = \{0\}$, we have $y - y' = z' - z = 0$, proving the claim. ■

This theorem is quite useful as it says that every element of H can be uniquely represented as the sum of an element of S and an element of $S^{\perp}$. We state this symbolically as

$$H = S \oplus S^{\perp}. \tag{2.19}$$

This allows us to define very important maps that we will use in detail later on in section 4.3.

Definition 2.26. Let S be a closed subspace of a Hilbert space H. The map $P_S : H \to H$ that sends an element $x = y + z \in S \oplus S^{\perp}$ to y is called the *orthogonal projection* onto S.

Remark 2.27. Suppose $S \subseteq H$ is a closed subspace of a Hilbert space H and P_S is the orthogonal projection of H onto S. We then have the following properties.

(i) P_S is an operator with norm less than or equal to 1.

(ii) P_s is idempotent, i.e. $P_S^2 = P_S$.

(iii) The restriction of P_S to S is the identity operator.

Proof:

(i) Suppose x_1 and x_2 are elements of H with unique decompositions $y_1 + z_1$ and $y_2 + z_2$ in $S \oplus S^{\perp}$ respectively. We then have that for $\alpha \in \mathbb{C}$, $P_S(x_1 + \alpha x_2) = y_1 + \alpha y_2 = P_S(x_1) + \alpha P_S(y_2)$, thus showing P_S is linear. For boundedness, by the Pythagoreans theorem 2.7 we have that

$$\|P_S x_1\|^2 = \|y_1\|^2 = \|x_1\|^2 - \|z_1\|^2 \leq \|x_1\|^2,$$

which shows that $\|P_S\| \leq 1$. Moreover, for every $x \in S$ we have $P_S(x) = x$. It follows that if $S \neq \{0\}$, and thus P_S is not the zero operator, then $\|P_S\| = 1$.

Remarks (ii) and (iii) are trivial and follow directly from the definition. ■

We will return to the concept of orthogonal projections later in section 4.3, where they will be used for defining *projection valued measures* that will be used in the spectral theorem.

Corollary 2.27.1. *Suppose S is a closed subspace of a Hilbert space H. Then $S^{\perp\perp} = S$.*

Proof: If $x \in S$, then for every $z \in S^{\perp}$ we have $(x \mid z) = 0$ and so $x \in S^{\perp\perp}$. Thus $S \subseteq S^{\perp\perp}$.
For the other direction, suppose $x \in S^{\perp\perp}$. Since S is closed, by 2.25, $x = y + z$ for some $y \in S$ and some $z \in S^{\perp}$. Since $y \in S$, we must have $y \in S^{\perp\perp}$, and therefore $z = x - y \in S^{\perp\perp}$. Since $z \in S^{\perp}$ also and since $S^{\perp} \cap S^{\perp\perp} = \{0\}$, we must have $z = 0$, and thus $x = y \in S$, and so $S^{\perp\perp} \subseteq S$ is true also, completing the proof. ■

The special geometry of Hilbert space allows us to produce a very rigid notion of basis not available in general Banach spaces. We spend the last part of this section developing this notion of a Hilbert space basis.

Definition 2.28. Let $F \subseteq H$ be an orthonormal system in a Hilbert space H. We say that F is *maximal among the orthonormal subsets of H*, or simply *maximal* if whenever $G \subseteq H$ is another orthonormal subset of H such that $F \subseteq G$, then $F = G$.

Lemma 2.29. *Let $S \subseteq H$ be an orthonormal system in a Hilbert space H, and let $x \in H$. Then $(x \mid s) \neq 0$ for at most countably many $s \in S$.*

Proof: Let $S_x := \{s \in S \mid (x \mid s) \neq 0\}$ and for each $n \in \mathbb{N}$, let $S_x^n := \{s \in S \mid (x \mid s) \geq \frac{1}{n}\}$. It is easily seen that $S^x = \bigcup_{n=1}^{\infty} S_x^n$. We claim that each S_x^n is finite. Suppose we have k distinct elements in S_x^n, label them $s_1, s_2, \dots, s_k$. By Bessel's inequality (2.14) we have that

$$k\left(\frac{1}{n}\right)^2 = \sum_{i=1}^{k} |(x \mid s)|^2 \leq \|x\|^2.$$

Thus k is bounded by $n^2\|x\|^2$ and so each S_x^n has at most $n^2\|x\|^2$ elements. Each S_x^n is therefore finite, and since S_x is the countable union of finite sets, it is also countable. ■

Theorem 2.30. *Let S be a maximal orthonormal system for a Hilbert space H. Then for all $x \in S$ we have the following relation.*

$$x = \sum_{s \in S} (x \mid s)s \tag{2.20}$$

Proof: Let S be a such maximal orthonormal system for H. For each $x \in H$, let $S_x := \{s \in S \mid (x \mid s) \neq 0\}$. By 2.29, S_x is countable and so we can rewrite this set as $S = \{s_n \mid n \in \mathbb{N}\}$. Define a sequence $(y_n)_{n\in\mathbb{N}} \in H^{\mathbb{N}}$ by

$$y_n = \sum_{k=1}^{n} (x \mid s_n)s_n.$$

We claim that $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. For $m > n$ we have that

$$\|y_m - y_n\|^2 = \left\| \sum_{k=n}^{m} (x \mid s_k)s_k \right\|^2 = \sum_{k=n}^{m} |(x \mid s_k)|^2.$$

By Bessel's inequality (2.14) we know that the series of positive real numbers $\sum_{n=1}^{\infty} |(x \mid s_k)|^2$ is bounded by $\|x\|^2$, and is therefore convergent and hence Cauchy. Thus $(y_n)_{n\in\mathbb{N}}$ is also Cauchy and since H is complete it converges.
Let $y = \sum_{n=1}^{\infty} (x \mid s_n)s_n$. We make use of the continuity of the inner product in the following equations.

$$\begin{aligned}
(y \mid s_k) &= \left(\lim_{n\to\infty} \sum_{j=1}^{n} (x \mid s_j)s_j \mid s_k \right) \\
&= \lim_{n\to\infty} \left(\sum_{j=1}^{n} (x \mid s_j)s_j \mid s_k \right) \\
&= \lim_{n\to\infty} (x \mid s_k) = (x \mid s_k).
\end{aligned}$$

Thus $(x - y \mid s_k) = 0$ for all $k \in \mathbb{N}$. Repeating this argument for the $s \in S$ such that $(x \mid s) = 0$ also gives $(x - y \mid s) = 0$, and so the vector $x - y$ is orthogonal to S. If x were not equal to y, the set $S \cup \{\frac{x-y}{\|x-\|}\}$ would be an orthonormal system in H strictly containing S, contradicting our maximality assumption. It follows that $x = y$. ■
The above theorem is very useful as it gives us an explicit description for any vector in a Hilbert space in terms of a maximal orthonormal subset. It therefore motivates the following definition.

Definition 2.31. We say a subset S of a Hilbert space H is an *orthonormal basis*, or *Hilbert space basis*, for the Hilbert space H, if it is a maximal orthonormal system.

Theorem 2.32. *Every Hilbert space has an orthonormal basis.*

Proof: Let H be a Hilbert space and let $\mathfrak{C}$ be the collection of all orthonormal subsets of H. Since $\emptyset$ is an orthonormal system, $\mathfrak{C}$ is non-empty. For any chain $A \subseteq C$, we have $\bigcup A$ as an upper bound for A that is also an orthonormal system. By Zorn's lemma, a maximal orthonormal system in $\mathfrak{C}$ exists. ■

Example 2.33. Consider the Hilbert space of absolutely square summable sequences ℓ^2. An orthonormal basis is given by the collection $\{e_n \mid n \in \mathbb{N}\}$ where $e_n = \underbrace{(0, \dots 0, 1, 0, \dots)}_{n^{\text{th}}\,\textit{position}}$.

By theorem 2.30 we can write any sequence $(x_n)_{n\in\mathbb{N}} \in \ell^2$ in terms of this basis by

$$(x_n)_{n\in\mathbb{N}} = \sum_{n=1}^{\infty}((x_n)_{n\in\mathbb{N}} \mid e_n)e_n.$$

Corollary 2.33.1. *Suppose $S \subseteq H$ is an orthonormal basis for the Hilbert space H. Then for each $x, y \in H$ we have that*

$$(x \mid y) = \sum_{s\in S}(x \mid s)\overline{(y \mid s)} \tag{2.21}$$

and

$$\|x\|^2 = \sum_{s\in S}|(x \mid s)|^2. \tag{2.22}$$

Proof: Let $x, y \in H$ and let $S_{x,y} = \{s \in S \mid (x \mid s) \neq 0 \text{ or } (y \mid s) \neq 0\} = S_x \cup S_y$. By 2.29, $S_{x,y}$ is countable. As such, list the elements as $S_{x,y} = \{s_n \mid n \in \mathbb{N}\} \subseteq S$. Let $x = \lim_{n\to\infty}\sum_{j=1}^{n}(x \mid s_j)s_j$ and $y = \lim_{n\to\infty}\sum_{k=1}^{n}(y \mid s_k)s_k$. By continuity of the inner product, we have the following.

$$\begin{aligned}
(x \mid y) &= \lim_{n\to\infty}\left(\sum_{j=1}^{n}(x \mid s_j)s_j \,\middle|\, \sum_{k=1}^{n}(y \mid s_k)s_k\right) \\
&= \lim_{n\to\infty}\sum_{j,k=1}^{n}(x \mid s_j)\overline{(y \mid s_k)}(s_j \mid s_k) \\
&= \lim_{n\to\infty}\sum_{j=1}^{n}(x \mid s_j)\overline{(y \mid s_j)}
\end{aligned}$$

Replacing y with x in the previous sequence of equations gives the second result. ■

From linear algebra, a similar notion of basis for a general vector space can be defined as a maximal linearly independent set. This is known as a *Hamel basis*, and can equivalently be described as follows.

Definition 2.34. Let V be a vector space. A collection $F := \{e_i \mid i \in I\} \subseteq V$ is a *Hamel basis* for V if the following two conditions hold.

(i) Any vector $x \in V$ can be written as a finite linear combination of the elements in F.

(ii) If $\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n = 0$ for some finite sum of elements $e_i \in F$, then $\alpha_i = 0$, $1 \leq i \leq n$.

Similarly to 2.32, using Zorn's lemma it can be shown that every vector space has a basis, for a full proof of this see theorem 1.27 in [7]. In fact, it was shown in [4] that if every vector space has a Hamel basis, then Zorn's Lemma is true.
For Banach spaces, however, the notion of Hamel basis is not particularly apt, as the next result shows.

Theorem 2.35. *Suppose E is an infinite dimensional Banach space. Then every Hamel basis for E is uncountable.*

Proof: Let E be an infinite dimensional Banach space, and aiming towards contradiction suppose that $(e_n)_{n\in\mathbb{N}}$ is a countable Hamel basis for E. By the Baire category theorem, E is of second category A.10.
For $n \in \mathbb{N}$, let $F_n = \text{span}\{e_1, \dots, e_n\}$. This is a finite dimensional normed space, hence closed. Clearly $E = \bigcup_{n=1}^{\infty} F_n$, and so by the Baire category theorem $\bigcup_{n=1}^{\infty} (F_n)^o$ is dense in E. Thus there exists some $n_0 \in \mathbb{N}$ such that $F_{n_0}{}^\circ \neq \emptyset$. Let $x \in F_{n_0}$. Since the interior is open, there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq F_{n_0}$.
Consider the vector e_{n_0+1} and take $\delta > 0$ large enough so that $\|\frac{1}{\delta} e_{n_0+1}\| < \varepsilon$, i.e, $\frac{1}{\delta} e_{n_0+1} \in B(0, \varepsilon)$. Thus $x + \frac{1}{\delta} e_{n_0+1} \in B(x, \varepsilon)$. But $x + \frac{1}{\delta} e_{n_0+1} \notin F_{n_0}$, contradicting $B(x, \varepsilon) \subseteq F_{n_0}$. Thus, no such countable Hamel basis exists. ■

We have seen in this section that every Hilbert space has an orthonormal basis. A countable basis is obviously easier to work with, and in fact, we can characterise those Hilbert spaces that have a countable basis.

Theorem 2.36. *A Hilbert space H is separable if and only if it has a countable orthonormal basis.*

Proof: Suppose H is separable. Let S be an orthonormal basis for H. Suppose $s_1, s_2 \in S$ are distinct. Then $d(s_1, s_2) = \|s_1 - s_2\| = \sqrt{(s_1 - s_2 \mid s_1 - s_2)} = \sqrt{2}$. Thus S is discrete when viewed as a metric space.

Since H is separable, so too is S. Thus S has a countable dense subset. But S is the only subset of itself that is dense in S, since S has the discrete metric, and so S must be countable.
Conversely, suppose H has a countable orthonormal basis $S = \{s_n \mid n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, consider the set

$$D_n = \left\{ \sum_{j=1}^{n} (q_j + ip_j)s_j \mid q_j, p_j \in \mathbb{Q} \text{ for } 1 \leq j \leq n \right\}.$$

Each D_n is countable and their union $D = \bigcup_{n=1}^{\infty} D_n$ is countable and dense in H. Moreover, the closure of D is the linear span of $s_1, s_2, \ldots, s_n$, and so the closure of D includes all finite linear combinations of the form $\sum_{j=1}^{n} z_j s_j$ where $z_j \in \mathbb{C}$. By 2.30 each $x \in H$ is the limit of such a finite linear combination. Hence the closure of D is all of H. As D is countable, H is separable.

■

2.4 Isomorphisms of Hilbert Spaces

In this section we will develop a notion of similarity between Hilbert spaces. In particular we discuss some different types of isomorphisms between Hilbert spaces and related results.

Definition 2.37. Let H and K be Hilbert Spaces. We say that H and K are *isometrically isomorphic*, and denote this by HK, if there is a surjection $T : H \to K$ such that for all $x, y \in H$ we have

$$(Tx \mid Ty) = (x \mid y). \tag{2.23}$$

Remark 2.38. If $T : H \to K$ is an isometric isomorphism between the Hilbert spaces H and K, it is also linear.
To see this let $x, y \in H$ and let $\alpha \in \mathbb{C}$. Since T is surjective there exists $z \in H$ such that $Tz = T(\alpha x + y) - (\alpha Tx + Ty)$. Then consider the following,

$$\begin{aligned}(Tz \mid Tz) &= (T(\alpha x + y) - (\alpha Tx + Ty) \mid Tz) \\ &= (T(\alpha x + y) \mid Tz) - [\alpha(Tx \mid Tz) + (Ty \mid Tz)] \\ &= (\alpha x + y \mid z) - [\alpha(x \mid z) + (y \mid z)] \\ &= 0\end{aligned}$$

Since the inner product is positive definite, it follows that $T(\alpha x + y) = \alpha Tx + y$ and so T is linear.

Proposition 2.39. *Suppose H and K are Hilbert spaces and $T : H \to K$ is a surjective linear map. Then T is an isometric isomorphism in the sense detailed above, if and only if it is an isometry.*

Proof: Suppose first that T is an isometric isomorphism. Then for all $x, y \in H$ we have $(Tx \mid Ty) = (x \mid y)$. Thus $\|Tx\|^2 = (Tx \mid Tx) = (x \mid x) = \|x\|^2$ and so T is an isometry.
Now suppose that T is an isometry. If $x, y \in H$ and $\alpha \in \mathbb{C}$, we have that $\|x + \alpha y\|^2 = \|Tx + \alpha Ty\|^2$. By the polar identity, (2.6), we therefore have that,

$$\|x\|^2 + 2\,\mathrm{Re}\,[\overline{\alpha}(x \mid y)] + |\alpha|^2\|y\|^2 = \|Tx\|^2 + 2\,\mathrm{Re}\,[\overline{\alpha}(Tx \mid Ty)] + |\alpha|^2\|Ty\|^2.$$

Since T is an isometry we have that $\|Tx\| = \|x\|$ and $\|Ty\| = \|y\|$, reducing the last equation to

$$\mathrm{Re}\,[\overline{\alpha}(x \mid y)] = \mathrm{Re}\,[\overline{\alpha}(Tx \mid Ty)]\,.$$

This is true for all $\alpha \in \mathbb{C}$. Setting $\alpha = 1$ gives $\mathrm{Re}\,[(x \mid y)] = \mathrm{Re}\,[(Tx \mid Ty)]$, and similarly setting $\alpha = i$ gives

$$\mathrm{Im}[(x \mid y)] = \mathrm{Re}\,[-i(x \mid y)] = \mathrm{Re}\,[-i(Tx \mid Ty)] = \mathrm{Im}[(Tx \mid Ty)].$$

Thus $(x \mid y)$ and $(Tx \mid Ty)$ have the same real and imaginary parts, and are the same. ■

Remark 2.40. Suppose $T : H \to K$ is an isometry. Then $\|T\| = 1$. This follows from the fact that

$$\|T\| = \sup\{\|Tx\| : x \in E_1\} = \sup\{\|x\| : x \in E_1\} = 1$$

Remark 2.41. If $T : H \to K$ is an isometric isomorphism then $T^{-1} : K \to H$ is also an isometric isomorphism.
To see this, note first that as isometries are injective, T is bijective and linear. Thus T^{-1} exists and is also bijective and linear. To see it is an isometry, let $y, y' \in K$. Then there exists $x, x' \in H$ respectively such that $T^{-1}(y) = x$ and $T^{-1}(y') = x'$, by bijectivity of T. We then have the following to show T^{-1} is an isometric isomorphism as claimed,

$$(T^{-1}y \mid T^{-1}y') = (x \mid x') = (Tx \mid Tx) = (y \mid y').$$

It is easy to show that this concept of isomorphism is an equivalence relation on any collection of Hilbert spaces. It is also the most appropriate notion of equivalence, since an inner product is the essential ingredient for a Hilbert space and isomorphic Hilbert spaces have the "same" inner product.
There is however another important notion of isomorphism between Hilbert spaces that connects better with the more general setting of Banach spaces. We detail this now.

Definition 2.42. Let H and K be Hilbert spaces. We say that a linear map $T : H \to K$ is a *topological isomorphism*, if it is bijective, and both T and T^{-1} are bounded.

It is easy to see that an isometric isomorphism is always a topological isomorphism. If $T : H \to K$ is an isomorphism in the isometric sense, remark 2.41 says that T^{-1} is also an isometric isomorphism. By remark 2.40, both T and T^{-1} have norm 1 and so the claims holds.

Theorem 2.43. *Suppose H is an infinite dimensional separable Hilbert Space. Then H is isometrically isomorphic to ℓ^2.*

Proof: Let H be an infinite dimensional separable Hilbert space. By theorem 2.36, there exists one countable orthonormal basis $(e_n)_{n\in\mathbb{N}}$ for H. Let $T : H \to \ell^2$ be defined as follows, for $x \in H$:

$$T(x) := [(e_n \mid x)]_{n\in\mathbb{N}}.$$

So T maps elements in the Hilbert space to a sequence generated by the inner product with the orthonormal basis. We must check that this map is a well defined isometric isomorphism. To see it is well defined, use Bessel's inequality (2.14) to get that

$$\|T(x)\|^2 = \sum_{n=1}^{\infty} (e_n \mid x)^2 \leq \|x\|^2 < \infty.$$

To see that T is linear let $x, y \in H$ and $\alpha \in \mathbb{C}$. Then,

$$T(x + \alpha y) = [(e_n \mid x + \alpha y)]_{n\in\mathbb{N}} = [(e_n \mid x)]_{n\in\mathbb{N}} + \alpha[(e_n \mid y)]_{n\in\mathbb{N}} = T(x) + \alpha T(y).$$

By 2.33.1, we see that $(Tx \mid Ty) = (x \mid y)$ and so T preserves the inner product. Lastly we must show that T is surjective. Let $(a_n)_{n\in\mathbb{N}} \in \ell^2$. Consider the sum $x_a = \sum_{n=1}^{\infty} a_n s_n$. By the generalised Pythagorean theorem 2.19 we get that

$$\|x_a\| = \sum_{n=1}^{\infty} |a_n|^2 < \infty,$$

and so x_a converges. Then note that

$$T(x_a) = [(e_n \mid x_a)]_{n\in\mathbb{N}} = \left[\left(e_n \,\Big|\, \sum_{k=1}^{\infty} a_k s_k\right)\right]_{n\in\mathbb{N}} = [a_n \|e_n\|^2]_{n\in\mathbb{N}} = [a_n]_{n\in\mathbb{N}}.$$

Hence $Tx_a = [a_n]_{n\in\mathbb{N}}$ and the proof is complete.

$\blacksquare$

2.5 Linear Functionals and Weak Topologies

The first main result of this section, the Riesz representation theorem, is fundamental in the study of Hilbert spaces, and simplifies the study of the dual space of a Hilbert space to the space itself. It allows also for simplified proofs of other results in functional analysis such as the Hahn–Banach theorem and others. We will first prove a useful lemma.

Lemma 2.44. *Suppose φ is a bounded linear functional on an inner product space E. Then* $\dim\left(\ker(\varphi)^{\perp}\right) \leq 1$.

Proof: If φ is the zero functional then $\ker(\varphi) = E$. Since $E^{\perp} = \{0\}$ we have $\dim\left(\ker(\varphi)\right)^{\perp} = 0$ and so the inequality is true in this case.
Suppose now that φ is not the zero operator. Since $\ker(\varphi) = \varphi^{-1}\left(\{0\}\right)$, $\{0\}$ is closed, and φ is continuous, it follows that $\ker(\varphi)$ is a closed subspace of E.

Thus by theorem 2.25, we can decompose E into $E = \ker(\varphi) + \ker(\varphi)^{\perp}$. Since $\ker(\varphi) \neq E$, the dimension of $(\ker(\varphi))^{\perp}$ is at least 1. To see that it is not more than 1, suppose we have two non-zero vectors x and y in $(\ker(\varphi))^{\perp}$. Since $\varphi x \neq 0$ and $\varphi y \neq 0$, there exists $\alpha \in \mathbb{C}$ such that $\alpha \varphi x + \varphi y = 0$, and by linearity of φ we have $\varphi(x + \alpha y) = 0$. Thus $x + \alpha y$ is in the kernel of φ.
On the other hand, since $(\ker(\varphi))^{\perp}$ is a subspace of E, $x + \alpha y \in (\ker(\varphi))^{\perp}$ also. Since $\ker(\varphi) \cap (\ker(\varphi))^{\perp} = \{0\}$, it follows that $x + \alpha y = 0$. Thus x and y are linearly dependent, and so the dimension of $(\ker(\varphi))^{\perp}$ cannot be two or more. We conclude that $\dim (\ker(\varphi))^{\perp} = 1$. ■

Theorem 2.45 (Riesz Representation Theorem)**.** *Let H be a Hilbert space. For every $\varphi \in H'$, there exists a unique $x_\varphi \in H$ called the Riesz Representation of φ, such that $\varphi(x) = (x \mid x_\varphi)$, for all $x \in H$. Furthermore, the norm of x_φ coincides with that of φ, that is, $\|\varphi\|_{H'} = \|x_\varphi\|_H$.*

Proof: If φ is the zero functional, then $x_\varphi = 0$ clearly satisfies the criteria.
Otherwise assume that φ is a non-zero functional, that is, there exists $x \in H$ such that $\varphi(x) \neq 0$. By lemma 2.44, $\dim (\ker(\varphi))^{\perp} = 1$. Let $z_0 \in \ker(\varphi)^{\perp}$ be a unit vector. Then for every $x \in H$ we have the following,

$$\varphi(x) = \frac{\varphi(x)}{\varphi(z_0)} \varphi(z_0) = \varphi\left(\frac{\varphi(x)}{\varphi(z_0)} z_0\right).$$

Thus, $x - \frac{\varphi(x)}{\varphi(z_0)} z_0 \in \ker(\varphi)$. From this, and using the fact that $z_0 \in \ker(\varphi)^{\perp}$ we get that

$$0 = \left(x - \frac{\varphi(x)}{\varphi(z_0)} z_0 \mid z_0\right) = (x \mid z_0) - \frac{\varphi(x)}{\varphi(z_0)}.$$

It follows that $\varphi(x) = (x \mid \overline{\varphi(z_0)} z_0)$.Therefore, let $x_\varphi = \overline{\varphi(z_0)} z_0$ to get $\varphi(x) = (x \mid x_\varphi)$ for all $x \in H$.
For uniqueness, suppose there is another point $x' \in H$ such that $\varphi(x) = (x \mid x')$ for all $x \in H$. Then $(x \mid x_\varphi - x') = (x \mid x_\varphi) - (x \mid x') = \varphi(x) - \varphi(x) = 0$. Thus $(x_\varphi - x' \mid x_\varphi - x') = 0$ and so $x' = x_\varphi$.
For the last part, by the Cauchy–Schwarz inequality 2.8 we have that,

$$\|\varphi\| = \sup_{x \in H_1} |\varphi(x)| = \sup_{x \in H_1} |(x \mid x_\varphi)| \leq \sup_{x \in H_1} \|x\| \|x_\varphi\| = \|x_\varphi\|.$$

We also have

$$\|x_\varphi\|^2 = (x_\varphi \mid x_\varphi) = |\varphi(x_\varphi)| \leq \|\varphi\| \|x_\varphi\|.$$

Combining these two inequalities gives the desired result. ■

Corollary 2.45.1. *There is an antilinear bijective isometry between any Hilbert space H and its dual space H'.*

Proof: Let H be a Hilbert space and define $S : H \to H'$ as follows.
For $x \in H$ define $S(x) : H \to \mathbb{C}$ by $S(x)(y) := (y \mid x)$. For $x_1, x_2, y \in H$, $\alpha \in \mathbb{C}$, we see that

$$\begin{aligned} S(x_1 + \alpha x_2)(y) &= (y \mid x_1 + \alpha x_2) \\ &= (y \mid x_1) + \overline{\alpha}(y \mid x_2) \\ &= S(x_1)(y) + \overline{\alpha} S(x_2)(y). \end{aligned}$$

Since this holds for all $y \in H$ we have $S(x_1 + \alpha x_2) = S(x_1) + \overline{\alpha} S(x_2)$ and S is indeed antilinear.
To see that it is an isometry, by Cauchy–Schwarz we have that

$$\begin{aligned} \|Sx\| &= \sup\{|Sx(y) \mid y \in H_1\} \\ &= \sup\{|(y \mid x)| \mid y \in H_1\} \\ &\leq \|x\| \sup\{\|y\| \mid y \in H_1\} \\ &= \|x\|. \end{aligned}$$

For the reverse direction, since $y := \frac{x}{\|x\|}$ has norm 1 for $x \neq 0$, $S(x)(y) = \frac{1}{\|x\|}(x \mid x) = \|x\|$ is an element of the set $\{|S(x)(y)| : y \in H_1\}$ and so $\|x\| \leq \|Sx\|$ as desired. As a result S is also injective. Lastly for the surjectivity, if $\varphi \in H'$, the Riesz representation theorem gives a unique $x_\varphi \in H$ such that for all $x \in H$, $\varphi(x) = (x \mid x_\varphi) = S(x_\varphi)(x)$. Thus $S(x_\varphi) = \varphi$ and the claim holds. ■

Theorem 2.46. *Every Hilbert space is reflexive.*

Proof: By corollary 2.45.1 there is an anti-linear isometric bijection

$$S_1 : H \to H'$$

given by $S_1(x)(y) = (y \mid x)$. Continuing on with this, there is another anti-linear isometric bijection $S_2 : H' \to H''$ given by $S_2(f, g) = (x_g \mid x_f)$ where x_f, and x_g are the Riesz representatives of f and g respectively. Since the composition of two anti-linear maps is linear, the composition $S_2 \circ S_1 : H \to H^{**}$ is an isometric isomorphism. Lastly, note that $J = S_2 \circ S_1$, where J is the canonical embedding of H into its bidual H'' A.2, as for any $x \in H$ and any $f \in H^*$ we have $(S_2 \circ S_1)(x)(f) = (f \mid S_1 x) = f(x) = J(x)(f)$.

■

We next look at what it means for a sequence in a Hilbert space to converge weakly. This will allow for more general results to hold as we will see shortly.

Definition 2.47. A sequence $(x_n)_{n\in\mathbb{N}}$ of vectors in an inner product space E is called *weakly convergent* to a vector x in E if $\lim_{n\to\infty}(x_n \mid y) = (x \mid y)$ for all $y \in E$. This is denoted by $x_n \rightharpoonup x$.

In contrast, (x_n) converges to x strongly if $\lim_{n\to\infty} \|x_n - x\| = 0$. It's easy to see that strong convergence implies weak convergence. For suppose (x_n) converges to x strongly. By the Cauchy–Schwarz inequality we have $|(x_n - x \mid y)| \leq \|x_n - x\|\|y\| \to 0$. Thus $(x_n - x \mid y) \to 0$ which gives $(x_n \mid y) \to (x \mid y)$.

Example 2.48. Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal sequence in a Hilbert space H. For any $x \in H$ we have by Bessel's inequlaity 2.20 that the series $\sum_{n=1}^{\infty} |(x \mid e_n)|^2$ is convergent and hence $(e_n \mid x) \to 0 = (0 \mid x)$ as $n \to \infty$. Since x was arbitrary we conclude that e_n converges to 0 weakly.
Conversly, for all $n \neq m$ we have

$$\|e_n - e_m\|^2 = \|e_n\|^2 + \|e_m\|^2 = 2.$$

and so $(e_n)_{n\in\mathbb{N}}$ is not Cauchy, hence not strongly convergent.

Theorem 2.49. *Weak limits of weakly convergent sequences are unique.*

Proof: Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a Hilbert space H and suppose $x_n \rightharpoonup x$, and $x_n \rightharpoonup z$ also. Then for all $y \in H$ $(x_n \mid y) \to (x \mid y)$ and $(x_n \mid y) \to (z \mid y)$. Since limits are unique in $\mathbb{C}$, it follows that $(x \mid y) = (z \mid y)$. Hence $x - z \in H^{\perp} = \{0\}$. ∎

Theorem 2.50. *Let $(x_n)_{n\in\mathbb{N}}$ be a weakly convergent sequence in a Hilbert space H. Then $(x_n)_{n\in\mathbb{N}}$ is bounded.*

Proof: Let $(x_n)_{n\in\mathbb{N}}$ be a weakly convergent sequence in a Hilbert space H. For $n \in \mathbb{N}$, define $f_n : H \to \mathbb{C}$ by $f_n(x) := (x \mid x_n)$. Note that each f_n is linear and $\|f_n(x)\| \leq \|x_n\|\|x\|$ and so $\|f_n\| \leq \|x_n\|$ and in particular f_n is bounded. Since for every $x \in H$, the sequence $((x \mid x_n))_{n\in\mathbb{N}}$ converges, it is bounded. Thus there exists $M_x > 0$ such that $|f_n(x)| \leq M_x$ for all $n \in \mathbb{N}$.
By the Uniform Boundedness Principle A.22, there exists a constant M such that $\|f_n\| \leq M$ for all $n \in \mathbb{N}$. Note that $|f_n(x_n)| = |(x_n \mid x_n)| = \|x_n\|^2$. Consequently $\|f_n\| = \|x_n\|$, and so $\|x_n\| \leq M$ for all $n \in \mathbb{N}$. ∎

We conclude this section with two key results in functional analysis.

Theorem 2.51. *Let H be a Hilbert space. Then every bounded sequence in H has a sequentially weakly convergent subsequence.*

Proof: Let $(h_n)_{n\in\mathbb{N}}$ be a bounded sequence in H. Let

$$H_0 = \overline{\operatorname{span}\{h_n \mid n \in \mathbb{N}\}}$$

We note that H_0 is separable, as the following set

$$\left\{ \sum_{k=1}^{n} (q_k + ip_k)h_k \mid q_k, p_k \in \mathbb{Q}, n \in \mathbb{N}_0 \right\},$$

serves as a countable dense subset. For each $n \in \mathbb{N}$, let $f_n : H_0 \to \mathbb{C}$ be defined by $f_n(h) = (h \mid h_n)$. Observe each f_n is an element of H' and that $\|f_n\| \leq \|h_n\|$. By A.27, $(f_n)_{n\in\mathbb{N}}$ has a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ which converges weak$-^*$ to a limit $f_0 \in H_0$.
By the Riesz representation theorem 2.45,there exists $h_0 \in H_0$ such that
$f_0(h) = (h \mid h_0)$ for all $h \in H_0$. Thus for every $h \in H$ we have that $\lim_{k\to\infty}(h \mid h_{n_k}) = (h \mid h_0)$.
Let $P : H \to H_0$ be the projection map from H onto H_0, that is, if $x = y + z$ where $y \in H_0$ and $z \in {H_0}^{\perp}$, then $P(x) = y$. For each $k \in \mathbb{N}$ we have that

$$((\mathrm{id} - P)(h) \mid h_{n_k}) = ((\mathrm{id} - P)(h) \mid h_0).$$

hence for each $h \in H$ we have that $\lim_{k\to\infty}(h_{n_k} \mid h) = (h_0 \mid h)$. Thus $(h_{n_k})_{k\in\mathbb{N}}$ converges weakly to $h_0 \in H$, proving the claim. ∎

The previous theorem shows that the closed unit ball of any Hilbert space is sequentially weakly compact. In the more general setting, this holds only for separable Banach spaces. However, the proof is far more intricate and involves appealing to theorems such as A.26 and others.

Our last result in this section, the Hahn–Banach theorem for Hilbert spaces, is particularly nice in this setting as it does not require the use of Zorn's lemma. Contrastingly, Zorn's Lemma is required in the wider Banach space setting. We first prove a useful lemma.

Lemma 2.52. *Suppose E_0 and E are normed spaces, and E_0 is dense in E. Let F be a Banach space and suppose $f_0 : E_0 \to F$ be a bounded linear operator. Then f_0 may be uniquely extended to a bounded linear operator $f : E \to F$ such that f_0 and f have the same norm.*

Proof: By denseness of E_0 in E, for each $x \in E$ there exists a sequence $(x_n)_{n\in\mathbb{N}} \in E_0^{\mathbb{N}}$ such that $\lim_{n\to\infty} x_n = x$. Hence $(x_n)_{n\in\mathbb{N}}$ is Cauchy, and since f_0 is bounded and linear we have that

$$\begin{aligned} \|f_0 x_n - f_0 x_m\| &= \|f_0(x_n - x_m)\| \\ &\leq \|f_0\|\|x_n - x_m\| \to 0. \end{aligned} \tag{2.24}$$

Thus, $(f_0 x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and hence by completeness of F, there exists $y \in F$ such that $\lim_{n\to\infty} f_0 x_n = y$. If there is another sequence $(x'_n)_{n\in\mathbb{N}} \in E_0^{\mathbb{N}}$ such that $\lim_{n\to\infty} x'_n = x$, then $\lim_{n\to\infty}(x'_n - x_n) = 0$, hence $\lim_{n\to\infty}(f_0 x'_n - f_0 x_n) = 0$ also. Thus, our choice of sequence converging to x is immaterial, and so we can umambiguously define $f : E \to F$ by $f(x) = \lim_{n\to\infty} f_0 x_n$, where $(x_n)_{n\in\mathbb{N}}$ is a sequence in E_0 converging to x.

It is clear that f is linear, as if we have $\alpha \in \mathbb{C}$ and $y \in E$ such that $(y_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} y_n = y$, then

$$f(x + \alpha y) = \lim_{n\to\infty}(f_0 x_n + \alpha f_0 y_n) = \lim_{n\to\infty} f_0 x_n + \alpha \lim_{n\to\infty} f_0 y_n = f(x) + \alpha f(y).$$

It is clear also that f is the only extension of f_0. For the norm properties, we have that

$$\begin{aligned} \|fx\| &= \left\|\lim_{n\to\infty} f_0 x_n\right\| \\ &= \lim_{n\to\infty} \|f_0 x_n\| \\ &\leq \lim_{n\to\infty} \|f_0\|\|x_n\| \\ &= \|f_0\|\|x_n\|. \end{aligned}$$

Thus, $\|f\| \leq \|f_0\|$. On the other hand, f is an extension of f_0 and so the norm of f_0 cannot be more than f, i.e. $\|f_0\| \leq \|f\|$. It follows that $\|f\| = \|f_0\|$. ∎

Theorem 2.53 (Hahn–Banach for Hilbert Spaces.)**.** *Let H be a Hilbert space and let $H_0 \subseteq H$ be a subspace. Suppose $f_0 \in H'_0$. Then there exists a unique $f \in H^*$ such that $f\,|_{H_0} = f_0$ and $\|f_0\| = \|f\|$.*

Proof: Without loss of generality we may assume H_0 is closed as a result of lemma 2.52. Then, being a closed subspace of a Hilbert space H, H_0 is also a Hilbert space and so by Riesz Representation 2.45, there exists $x_{f_0} \in H$ such that $f_0(x) = (x \mid x_{f_0})$ and $\|f_0\| = \|x_{f_0}\|$.
Define $f : H \to \mathbb{C}$ by $f(x) := (x \mid x_{f_0})$. Clearly f is linear and $\|f(x)\| = |(x \mid x_{f_0})| \leq$

$\|x_{f_0}\|\|x\| = \|f_0\|\|x\|$. Thus $\|f\| \leq \|f_0\|$. However f is clearly an extension of f_0 and so $\|f_0\| \leq \|f\|$ also. Thus $\|f\| = \|f_0\|$.

For uniqueness, let $\varphi \in H'$ be another extension of f_0 to H with the same norm. Using the Riesz representation once more 2.45, there exists $x_\varphi \in H$ such that $\varphi(x) = (x \mid x_\varphi)$ for all $x \in H$. Since $\|\varphi\| = \|f_0\|$ we have $\|f\| = \|\varphi\|$ which in turn means $\|x_{f_0}\| = \|x_\varphi\|$. Now since both f and φ are extensions of f_0, they agree on H_0. Thus for all $x \in H_0$ $f(x) = \varphi(x)$, i.e $(x \mid x_{f_0} - x_\varphi) = 0$. In particular $x_{f_0} \in H_0$ and so $x_{f_0} \perp (x_{f_0} - x_\varphi)$. Thus, by Pythagoras' theorem 2.7, we have the following,

$$\|x_{f_0}\| = \|x_\varphi\| = \|x_{f_0} + (x_\varphi - x_{f_0})\| = \sqrt{\|x_{f_0}\|^2 + \|x_{f_0} - x_\varphi\|^2}$$

This can only happen if $\|x_{f_0} - x_\varphi\| = 0$ and so we conclude that $x_{f_0} = x_\varphi$ which implies $f = \varphi$.

■

3 Operators

In this chapter, we develop the theory of bounded linear operators. A guiding source for this material is chapter 4 in [8]. Since we will only be dealing with bounded linear operators, we will simply call them operators from now on unless specified otherwise. To clarify notation, given operators A and B, we will denote their composition as AB rather than $A \circ B$.

3.1 Self-Adjoint Operators

To motivate this section, we begin with the following example. Let A be an operator on a Hilbert space H. For a fixed $y \in H$ the map $f : H \to \mathbb{C}$ given by $f(x) = (Ax \mid y)$ is a bounded linear functional on H. By the Riesz Representation theorem 2.45, there exists a unique $z \in H$ such that $f(x) = (x \mid z)$ for all $x \in H$, or equivalently, $(Ax \mid y) = (x \mid z)$. If we denote by A^* the map which to every $y \in H$ assigns that unique z, we then have $(Ax \mid y) = (x \mid A^*y)$ for all $x, y \in H$. We call this map the *adjoint* of A.

Definition 3.1. Let A be an operator on a Hilbert space H. The operator $A^* : H \to H$ defined by

$$(Ax \mid y) = (x \mid A^*y)$$

is called the *adjoint operator* of A.

Remark 3.2. The adjoint operator is also sometimes called the *Hermitian adjoint*, named in honour of the French mathematician Charles Hermite.

Theorem 3.3. *Let A be an operator on H. Then the adjoint A^* is also an operator. In addition, $\|A^*\| = \|A\|$ and $\|A^*A\| = \|A\|^2$.*

Proof: Let $x \in H$ be arbitrary and take a linear combination $\alpha y_1 + y_2 \in H$. We then compute the following.

$$\begin{aligned}(x \mid A^*(\alpha y_1 + y_2)) &= (Ax \mid \alpha y_1 + y_2) \\ &= \overline{\alpha}(Ax \mid y_1) + (Ax \mid y_2) \\ &= \overline{\alpha}(x \mid A^*y_1) + (x \mid A^*y_2) \\ &= (x \mid \alpha A^*y_1 + A^*y_2)\end{aligned}$$

This shows the linearity of A^*. For boundedness, we have the following.

$$\begin{aligned}\|A^*x\|^2 = (A^*x \mid A^*x) &= (A(A^*x) \mid x) \\ &\leq \|A(A^*x)\|\|x\| \leq \|A\|\|A^*x\|\|x\|.\end{aligned}$$

It follows that $\|A^*x\| \leq \|A\|\|x\|$, and so $\|A^*\| \leq \|A\|$. Interchanging the roles of A and A^* in the equations above shows also that $\|A\| \leq \|A^*\|$, giving $\|A\| = \|A^*\|$. Finally, take some $x \in H$ such that $\|x\| = 1$. We then have that

$$\|Ax\|^2 = (Ax \mid Ax) = (x \mid A^*Ax) \leq \|x\|\|A^*A\|\|x\| = \|A^*A\|.$$

Since

$$\|A^*A\| \leq \|A^*\|\|A\| = \|A\|^2,$$

altogether we get that $\|A^*A\| = \|A\|^2$, completing the assertion. ■

Let A and B be operators on H, with A^* and B^* being their respective adjoint operators, and let $\alpha \in \mathbb{C}$. The following properties arise as straightforward computations, whose proof we omit but can be found in [10], section 4.1.

(i) $(A + B)^* = A^* + B^*$

(ii) $(\alpha A)^* = \overline{\alpha} A^*$

(iii) $(A^*)^* = A$

(iv) $(AB)^* = B^*A^*$

(v) $\mathrm{id}^* = \mathrm{id}$

Our next result shows that we can characterise the kernel of the adjoint in terms of the image of the original operator.

Theorem 3.4. *Let A be an operator on H. Then $A(H)^\perp = \ker(A^*)$.*

Proof: Let $y \in A(H)^\perp$. For $x \in H$ we have $(x \mid A^*y) = (Ax \mid y) = 0$, which shows $y \in \ker(A^*)$. On the other hand, if $y \in \ker(A^*)$ we have $(Ax \mid y) = (x \mid A^*y) = 0$ for all $x \in H$, which shows $y \in A(H)^\perp$. ■

Since orthogonal complements are always closed (2.22), it follows that we can split the Hilbert space as follows,

$$\begin{aligned} H &= A(H) \oplus \left(A(H)^\perp\right)^\perp \\ &= \ker(A^*) \oplus A(H). \text{ (By corollary 2.27.1)} \end{aligned}$$

Next, we characterise the adjoint operator in the finite-dimensional setting.

Example 3.5. Let A be an operator on $\mathbb{C}^n$. We can represent A by an $n \times n$ matrix $[A_{ij}]$ with respect to the canonical basis $\{e_1, \ldots, e_n\}$. The matrix elements are determined by $A_{ij} = (Ae_j \mid e_i)$.
Let $[{A^*}_{ij}]$ be the matrix representing the adjoint A^*. The i, j^{th} entry ${A^*}_{ij}$ is then given as follows.

$${A^*}_{ij} = (A^* e_j \mid e_i) = \overline{(e_i \mid A^* e_j)} = \overline{(Ae_i \mid e_j)} = \overline{A_{ji}}.$$

It follows that the matrix representation for the adjoint A^* is the conjugate transpose of the matrix representation of A. In the case of a real vector space $\mathbb{R}^n$, this simplifies further to the transposed matrix.

This example illustrates that the adjoint of an operator and the operator itself are not necessarily the same. As a result, operators that are equal to their adjoint are given a special name.

Definition 3.6. Let A be an operator on H. A is called *self-adjoint* if $A = A^*$, i.e. $(Ax \mid y) = (x \mid Ay)$ for all $x, y \in H$.

Theorem 3.7. *Suppose T is an operator on a Hilbert space H. Then there exist unique self-adjoint operators A and B such that $T = A + iB$.*

Proof: Let T be an operator on H. Defined A and B as follows,

$$A = \frac{1}{2}(T + T^*) \text{ and } B = \frac{1}{2i}(T - T^*).$$

A routine check shows A and B are both self-adjoint, and that $T = A + iB$. Furthermore, for any $x, y \in H$ we have the following,

$$\begin{aligned}(Tx \mid y) &= ((A + iB)x \mid y) \\ &= (Ax \mid y) + i(Bx \mid y) \\ &= (x \mid Ay) + i(x \mid By) \\ &= (x \mid (A - iB)y) = (x \mid T^* y).\end{aligned}$$

Thus, $(Tx \mid y) = (x \mid T^* y)$. For uniqueness, suppose there is another decomposition $T = A' + iB'$, where A' and B' are also self-adjoint operators on H. We then have that $T = A - A' = iB' - iB$. Since $T^* = (A - A')^* = A - A' = T$ we have $T^* = T$. On the other hand, we also have $T^* = (iB' - iB)^* = -(iB' - iB) = -T$. It therefore follows that $T = -T$ and so $T = 0$, hence $A = A'$ and $B = B'$. This shows uniqueness, completing the proof. ■

Our next result gives an alternate method for calculating the operator norm of a self-adjoint operator.

Theorem 3.8. *Let A be a self-adjoint operator on a Hilbert space H. Then*

$$\|A\| = \sup\{|(Ax \mid x)| \mid x \in H_1\}.$$

Proof: Let $M_A := \sup\{|(Ax \mid x)| \mid x \in H_1\}$. By the Cauchy–Schwarz inequality 2.3 it is clear that that M_A exists and that $M_A \leq \|A\|$.
For the reverse, let $x \in H$ be such that $Ax \neq 0$. We then define $\alpha := \sqrt{\frac{\|Ax\|}{\|x\|}}$ and $z := \frac{Ax}{\alpha}$, and compute the following.

$$\begin{aligned}
\|Ax\|^2 &= (A(\alpha x) \mid z) \\
&= \frac{1}{4}\left[(A(\alpha x + z) \mid \alpha x + z) - (A(\alpha x - z) \mid \alpha x - z)\right] \\
&\leq \frac{1}{4} M_A \left[\|\alpha x + z\|^2 + \|\alpha x - z\|^2\right] \\
&= \frac{1}{2} M(\|\alpha x\|^2 + \|z\|^2) \\
&= \frac{1}{2} M_A \left[\alpha^2 \|x\|^2 + \frac{1}{\alpha^2}\|Ax\|^2\right] = M_A \|x\| \|Ax\|
\end{aligned}$$

Thus, $\|Ax\| \leq M\|x\|$ and so when paired with the earlier inequality we have $\|A\| = M_A$. ∎

Theorem 3.9. *Let A be a self-adjoint operator on a Hilbert space H. If $A(H)$ is dense in H, then A has an inverse map defined on the image $A(H)$ of A.*

Proof: By theorem 3.4 we split the space into $H = \ker(A^*) \oplus \overline{A(H)}$. Since $A(H)$ is dense in H, it follows that $\ker(A^*) = \{0\}$. As a result, $\ker(A) = \{0\}$ because A is self-adjoint, and so it is injective. Thus $A : H \to A(H)$ is bijective and a well-defined inverse map $A^{-1} : A(H) \to H$ exists. [1] ∎

3.2 Various Types of Operators

In this section, we explore and look at other types of operators, such as normal, unitary, projection, positive, compact, and finite rank operators.

Definition 3.10. Let A be an operator on a Hilbert space H. An operator B also defined on H is called an *inverse operator* of A, if $AB = \text{id}$ and $BA = \text{id}$.

[1] This inverse map may not necessarily be bounded, compare with corollary 3.24.1.

Remark 3.11. Suppose B_1 and B_2 are both inverse operators for A. Then $B_1 = B_1 \operatorname{id} = B_1 A B_2 = \operatorname{id} B_2 = B_2$. Thus the inverse operator is unique, and we denote it by A^{-1}.

Definition 3.12. An operator A is called *normal* if it commutes with its adjoint, that is, $AA^* = A^*A$. Moreover, if this product is the identity, the operator is called *unitary*, that is, $AA^* = \operatorname{id} = A^*A$.

Example 3.13. Let H be the Hilbert space of all sequences of complex sequences $x = (\dots, x_{-1}, x_0, x_1, x_2 \dots)$ such that $\sum_{n=-\infty}^{\infty} |x_n|^2 < \infty$. The inner product is defined by

$$(x \mid y) = \sum_{n=-\infty}^{\infty} x_n \overline{y_n}.$$

Define an operator $T \in B(H)$ by $T(x_n)_{n\in\mathbb{N}} = (x_{n-1})_{n\in\mathbb{N}}$, i.e. T sends x_n to x_{n-1} for each $n \in \mathbb{N}$. We then have that

$$(Tx \mid y) = \sum_{n=-\infty}^{\infty} x_{n-1}\overline{y_n} = \sum_{n=-\infty}^{\infty} x_n \overline{y_{n+1}} = (x \mid T^{-1}y).$$

It follows that $T^* = T^{-1}$, hence T is a unitary operator.

Theorem 3.14. *An operator A is normal if and only if $\|Ax\| = \|A^*x\|$ for all $x \in H$.*

Proof: We have that

$$(A^*Ax \mid x) = (Ax \mid Ax) = \|Ax\|^2.$$

Since A is normal, we also have

$$(A^*Ax \mid x) = (AA^*x \mid x) = (A^*x \mid A^*x) = \|A^*x\|^2.$$

Thus $\|Ax\| = \|A^*x\|$.
For the reverse, suppose $\|Ax\| = \|A^*x\|$. By the above argument we have $(AA^*x \mid x) = (A^*Ax \mid x)$. By corollary A.40, it follows that $A^*A = AA^*$, and so A is normal. ∎

Theorem 3.15. *An operator A defined on a Hilbert space H is isometric if and only if $A^*A = id$.*

Proof: If A is isometric, then for every $x \in H$ we have $\|Ax\|^2 = \|x\|^2$, hence $(A^*Ax \mid x) = (Ax \mid Ax) = \|x\|^2$ for all $x \in H$. By A.40, $A^*A = \operatorname{id}$.
Conversely, if $A^*A = id$, then $\|Ax\| = \sqrt{(Ax \mid Ax)} = \sqrt{(A^*Ax \mid x)} = \sqrt{(x \mid x)} = \|x\|$. ∎

Theorem 3.16. *An operator A is unitary if and only if it is invertible and $A^{-1} = A^*$.*

Proof: If A is invertible and $A^{-1} = A^*$, then $A^*A = A^{-1}A = id = AA^{-1} = AA^*$. ■

3.3 Compact Operators

We devote this section entirely to the study of compact operators as they form one of the most important classes of operators.

Definition 3.17. An operator A on a Hilbert space H is called a *compact operator* if $A(H_1)$ is precompact, where H_1 denotes the unit ball.

Remark 3.18. There are other equivalent definitions for an operator $A \in B(H)$ to be compact, and we will make use of two of them. Let $A \in B(H)$. Then the following are equivalent.

(i) A is compact.

(ii) For any bounded subset $S \subseteq H$, $A(S)$ is precompact.

(iii) For any bounded sequence $(x_n)_{n\in\mathbb{N}}$ in H, the sequence $(Ax_n)_{n\in\mathbb{N}}$ contains a convergent subsequence.

Proof: $(i) \iff (ii)$. Multiplication by a scalar is a linear homeomorphism, thus A is compact if and only $\overline{AH_1}$ is compact if and only if $n\overline{A(H_1)} = \overline{A(nH_1)}$ is compact.
$(ii) \implies (iii)$. Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in H. Then $S := \{x_n \mid n \in \mathbb{N}\}$ is a bounded subset in H, and so $\overline{A(S)}$ is compact, meaning $(Ax_n)_{n\in\mathbb{N}}$ admits a convergent subsequence with limit in H.
$(iii) \implies (ii)$. Let $S \subseteq H$ be bounded and let $(y_n)_{n\in\mathbb{N}} \in \overline{A(S)}$. Since S is bounded and A is an operator, $(y_n)_{n\in\mathbb{N}}$ is bounded. For each $n \in \mathbb{N}$ there exists $x_n \in S$ such that

$$\|y_n - Ax_n\| \leq 2^{-n}. \tag{3.1}$$

Since $(y_n)_{n\in\mathbb{N}}$ is bounded, so is $(Ax_n)_{n\in\mathbb{N}}$. By hypothesis, there exists a convergent subsequence $(Ax_{n_k})_{k\in\mathbb{N}}$, and by 3.1, $\lim_{k\to\infty} y_{n_k} = \lim_{k\to\infty} Ax_{n_k} \in H$. Thus $(y_n)_{n\in\mathbb{N}}$ has a convergent subsequence and as a result, $A(S)$ is precompact. ■

Example 3.19. Let $T : \mathbb{C}^n \to \mathbb{C}^n$ be a linear map, where $\mathbb{C}^n$ is equipped with the standard inner product. Then T is bounded by A.16. Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathbb{C}^n$. Then since T is an operator, $(Tx_n)_{n\in\mathbb{N}}$ is a bounded sequence. By the Bolzano–Weierstrass theorem, $(Ax_n)_{n\in\mathbb{N}}$ has a convergent subsequence, thus T is compact.

Example 3.20. Not all operators are compact. Take the identity map id on any Hilbert space H and let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal sequence in H. Then $(e_n)_{n\in\mathbb{N}}$ is a bounded sequence in H, yet $(\mathrm{id}\, e_n)_{n\in\mathbb{N}} = (e_n)_{n\in\mathbb{N}}$ has no convergent subsequence. To see this, suppose n and m are distinct natural numbers. We then have that

$$\begin{aligned}\|e_n - e_m\| &= \sqrt{(e_n - e_m \mid e_n - e_m)} \\ &= \sqrt{(e_n \mid e_n) - (e_n \mid e_m) - (e_m \mid e_n) + (e_m \mid e_m)} \\ &= \sqrt{1 - 0 - 0 + 1} = \sqrt{2}.\end{aligned}$$

Thus, the distance between any two distinct points in the sequence cannot be made arbitrarily small and no convergent subsequence exists.

Example 3.21. Let H be a Hilbert space. Given fixed elements y and z in H we can create a compact operator in the following way. Define $T : H \to H$ by

$$T(x) := (x \mid y)z.$$

That T is an operator is easy to check. To see that is compact, let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in H, so there is $K > 0$ such that $\|x_n\| \le K$ for all $n \in \mathbb{N}$. By Cauchy–Schwarz we know that $|(x_n \mid y)| \le \|x_n\|\|y\| \le M\|y\|$, and so by Bolzano–Weierstrass, $((x_n \mid y))_{n\in\mathbb{N}}$ contains a convergent subsequence $((x_{n_k} \mid y))_{k\in\mathbb{N}}$, whose limit we denote by l. Then $Tx_{k_n} = (x_{k_n} \mid y)z \to lz$. Thus T is compact.

Example 3.22. Let $S \subseteq H$ be a finite-dimensional subspace of a Hilbert space H. By A.17 S is closed and thus by 2.25 we can decompose H into $S \oplus S^{\perp}$. The projection operator $P_S : H \to H$, given by $P_S(x) = y$ where $x = y + z$ is the orthogonal decomposition, is compact. This is quite easy to see, as for a bounded sequence $(x_n)_{n\in\mathbb{N}} \in H^{\mathbb{N}}$, its decomposition is also bounded.

Compact operators upgrade the convergence of weakly convergent sequences to strong convergence, as our next result shows.

Theorem 3.23. *Let $(x_n)_{n\in\mathbb{N}}$ be a weakly convergent sequence with weak limit x in a Hilbert space H. For every compact operator A on H, the image sequence $(Ax_n)_{n\in\mathbb{N}}$ converges strongly to Ax. That is if $x_n \rightharpoonup x$, then $Ax_n \to Ax$.*

Proof: Since $Ax_n \rightharpoonup x$, for all $y \in H$ we have the following.

$$(Ax_n \mid y) = (x_n \mid A^*y) \to (x \mid A^*y) = (Ax \mid y).$$

Thus Ax_n converges to Ax weakly. Since weak limits are unique (2.49) and strong convergence implies weak convergence, the only possible strong limit for $(Ax_n)_{n\in\mathbb{N}}$ is Ax.

To show that $\lim_{n\to\infty} Ax_n = Ax$, suppose otherwise, aiming towards contradiction. Then there is a subsequence $(Ax_{n_k})_{k\in\mathbb{N}}$ of $(Ax_n)_{n\in\mathbb{N}}$ such that

$$\|Ax_{n_k} - Ax\| \geq \delta \text{ for all } k \in \mathbb{N} \text{ and some } \delta > 0.$$

Since $(x_n)_{n\in\mathbb{N}}$ is bounded by 2.50 and A is compact, there is a subsequence $(Ax_{n_{k_l}})_{l\in\mathbb{N}}$ of $(Ax_{n_k})_{k\in\mathbb{N}}$ for which $\lim_{l\to\infty} Ax_{n_{k_l}} = y \in H$. Since $Ax_{n_{k_l}} \rightharpoonup Ax$ as $l \to \infty$, we must have $y = Ax$, which is impossible by construction of Ax_{n_k}, completing the contradiction. Hence $\lim_{n\to\infty} Ax_n = Ax$. ■

Example 3.24. Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal sequence in a Hilbert space H. From example 2.48 we know that $(e_n)_{n\in\mathbb{N}}$ converges to 0 weakly. Hence by theorem 3.23, we conclude that $\lim_{n\to\infty} Ae_n = 0$ for all compact operators $A \in B(H)$.

Corollary 3.24.1. *Suppose H is an infinite-dimensional, separable Hilbert space and $A \in B(H)$ is a compact operator. Suppose furthermore that A has an inverse map A^{-1} defined on $A(H)$. Then A^{-1} is unbounded.*

Proof: Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis for H which exists by 2.36. Then $\lim_{n\to\infty} Ae_n = 0$ by example 3.24. On the other hand, $\|A^{-1}(Ae_n)\| = \|e_n\| = 1$, hence $\lim_{n\to\infty} A^{-1}(Ae_n) \neq 0$ and so A^{-1} is not continuous. ■

Definition 3.25. We denote by $\mathcal{K}(H)$ the set of all compact operators on a Hilbert space H.

$\mathcal{K}(H)$ is a vector space over $\mathbb{C}$ when equipped with pointwise addition of operators, a proof of this can be found in [14], 7.14, page 214.

Theorem 3.26. *Let $A \in \mathcal{K}(H)$ and $B \in B(H)$. Then AB and BA are both in $\mathcal{K}(H)$.*

Proof: Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in H. Since B is bounded the sequence $(Bx_n)_{n\in\mathbb{N}}$ is also bounded. Then since A is compact, the sequence $(ABx_n)_{n\in\mathbb{N}}$ contains a convergent subsequence. Thus AB is compact. As for BA, by compactness of A the sequence $(Ax_n)_{n\in\mathbb{N}}$ has a convergent subsequence, call it $(Ax_{n_K})_{k\in\mathbb{N}}$. Then by boundedness of B, the sequence $(BAx_{n_k})_{k\in\mathbb{N}}$ converges. ■

Definition 3.27. We say an operator $A \in B(H)$ is *finite rank* if the dimension of $A(H)$ is finite[2]. We denote by $\mathcal{F}(H)$ the collection of all finite rank operators on H.

[2]Recall that *operator* for us includes the assumption of boundedness

Similarly to $\mathcal{K}(H)$, $\mathcal{F}(H)$ is also a $\mathbb{C}$-vector space when equipped with pointwise addition of operators, and $\mathcal{F}(H) \subseteq \mathcal{K}(H)$ as our next result shows.

Theorem 3.28. *Let A be a finite rank operator on a Hilbert space H. Then A is a compact operator.*

Proof: Let A be a finite rank operator on H, and so $A(H)$ is finite-dimensional. Furthermore, if $(x_n)_{n\in\mathbb{N}}$ is a bounded sequence in H, then $(Ax_n)_{n\in\mathbb{N}}$ is also bounded. This is straightforward as if $\|x_n\| \leq M$ for all $n \in \mathbb{N}$, we have

$$\|Ax_n\| \leq \|A\|M < \infty.$$

It follows by the Bolzano–Weierstrass theorem, page 51, [17], that $(Ax_n)_{n\in\mathbb{N}}$ has a convergent subsequence, hence A is compact. ■

Our next theorem shows that the set of all compact operators $\mathcal{K}(H)$ is closed in the set of all operators $B(H)$.

Theorem 3.29. *Let H be a Hilbert space and suppose $(A_n)_{n\in\mathbb{N}}$ is a sequence of compact operators in $B(H)$ converging to an operator $A \in B(H)$. Then A is also compact.*

Proof: Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in H. Let $M \in \mathbb{R}_+$ such that $\|x_n\| \leq M$ for all $n \in \mathbb{N}$. Since A_1 is compact, $(x_n)_{n\in\mathbb{N}}$ has a subsequence $(x_{1,n})_{n\in\mathbb{N}}$ such that $(A_1x_{1,n})_{n\in\mathbb{N}}$ is convergent. Similarly, by compactness of A_2, the sequence $(x_{1,n})_{n\in\mathbb{N}}$ contains a subsequence $(x_{2,n})_{n\in\mathbb{N}}$ such that $(A_2x_{2,n})_{n\in\mathbb{N}}$ is convergent. Carrying on in this manner, for $k \geq 2$ let $(x_{k,n})_{n\in\mathbb{N}}$ be a subsequence of $(x_{k-1,n})_{n\in\mathbb{N}}$ such that $(A_kx_{k,n})_{n\in\mathbb{N}}$ converges. By construction, for each $k \in \mathbb{N}$, $(A_kx_{n,n})_{n\in\mathbb{N}}$ converges. We show that the sequence $(Ax_{n,n})_{n\in\mathbb{N}}$ converges also.
Let $\varepsilon > 0$. Since $\lim_{n\to\infty} A_n = A$, there exists $k_0 \in \mathbb{N}$ such that $\|A_{k_0} - A\| < \frac{\varepsilon}{3M}$. Next, since $(A_{k_0}x_{n,n})_{n\in\mathbb{N}}$ converges, it is also a Cauchy sequence and thus there exists $j \in \mathbb{N}$ such that for all $n, m > j$ we have

$$\|A_{k_0}x_{n,n} - A_{k_0}x_{m,m}\| < \frac{\varepsilon}{3}.$$

Putting this together, for all $n, m > j$ we have the following.

$$\begin{aligned}\|Ax_{n,n} - Ax_{m,m}\| &\leq \|Ax_{n,n} - A_{k_0}x_{n,n}\| + \|A_{k_0}x_{n,n} - A_{k_0}x_{m,m}\| + \|A_{k_0}x_{m,m} - Ax_{m,m}\| \\ &\|A - A_{k_0}\|\|x_{n,n}\| + \|A_{k_0}x_{n,n} - A_{k_0}x_{m,m}\| + \|A - A_{k_0}\|\|x_{m,m}\| \\ &< \frac{\varepsilon}{3M}M + \frac{\varepsilon}{3} + \frac{\varepsilon}{3M}M = \varepsilon.\end{aligned}$$

Thus, $(Ax_{n,n})_{n\in\mathbb{N}}$ is Cauchy and so by the completeness of H, $(Ax_{n,n})_{n\in\mathbb{N}}$ converges. Hence A is compact.

■

By Theorem 3.3, we know that the adjoint A^* is always bounded if A is an operator. The natural question relating to compactness is whether this holds true also for $A \in \mathcal{K}(H)$. This leads to our final result in this section.

Theorem 3.30. *Let A be a compact operator on a Hilbert space H. Then the adjoint operator A^* is also compact.*

Proof: Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in H, and let $M \in \mathbb{R}_+$ be such that $\|x_n\| \leq M$ for all $n \in \mathbb{N}$. Define a sequence $(y_n)_{n\in\mathbb{N}}$ in H by $y_n := A^*x_n$. Since A^* is bounded, $(y_n)_{n\in\mathbb{N}}$ is bounded also. By compactness of A, $(y_n)_{n\in\mathbb{N}}$ contains a subsequence $(y_{n_k})_{k\in\ltimes}$ such that $(Ay_{n_k})_{k\in\mathbb{N}}$ converges. For all $n, m \in \mathbb{N}$ we have the following.

$$\begin{aligned}
\|y_{k_m} - y_{k_n}\|^2 &= \|A^*x_{k_m} - A^*x_{k_n}\|^2 \\
&= (A^*(x_{k_m} - x_{k_n}) \mid A^*(x_{k_m} - x_{k_n})) \\
&= (AA^*(x_{k_m} - x_{k_n}) \mid (x_{k_m} - x_{k_n})) \\
&\leq \|AA^*(x_{k_m} - x_{k_n})\|x_{k_m} - x_{k_n})\| \\
&\leq 2M\|Ay_{k_m} - Ay_{k_n}\| \to 0.
\end{aligned}$$

Thus $(A^*x_{n_k})_{n\in\mathbb{N}}$ is a Cauchy sequence and by the completeness of H converges, proving the compactness of A^*. ■

4 Spectral Theory

In this chapter, we will look to develop the spectral theorem for compact self-adjoint operators, before proving it for self-adjoint operators more generally, i.e. without the compactness assumption. Sections 1.2 and 2.5 of [13] are used, along with chapter 5 of [10]. Chapter 8 of [9] is also followed closely.

4.1 The Spectrum of an Operator.

To begin this section, we define and look at eigenvalues and eigenvectors of matrices. Let $A \in M_n(\mathbb{C})$ be an $n \times n$ matrix.

Definition 4.1. We say that an element $\lambda \in \mathbb{C}$ is an *eigenvalue of* A if there exists $x \in \mathbb{C}^n \setminus \{0\}$ such that $Ax = \lambda x$. In this case, x is called an *eigenvector corresponding to* λ.

Remark 4.2. It is not hard to see that if λ is an eigenvalue of A and x is a corresponding eigenvector, then any non-zero $y \in \text{span}(x)$ is also an eigenvector corresponding to λ. Thus we may sometimes refer to the subspace of all eigenvectors corresponding to λ as the *eigenspace* with respect to λ, and denote it by $\text{Eig}(\lambda)$.

Example 4.3. Consider the matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. To calculate the eigenvalues of A, we seek $\lambda \in \mathbb{C}\setminus\{0\}$ such that $A\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$. This reduces to solving the following system of equations,

$$\begin{cases} -y = \lambda x \\ x = \lambda y \end{cases}$$

This gives $\lambda^2 = -1$, so the two solutions are $\lambda = \pm i$. An easy calculation shows that $\text{Eig}(i) = \text{span}\left\{\begin{pmatrix} 1 \\ -i \end{pmatrix}\right\}$ and similarly $\text{Eig}(i) = \text{span}\left\{\begin{pmatrix} 1 \\ i \end{pmatrix}\right\}$.

Example 4.4. As an example of an operator without any eigenvalues, consider the right shift operator $A : \ell^2 \to \ell^2$ that sends a sequence $(x_1, x_2, x_3, \dots)$ to $(0, x_1, x_2, x_3, \dots)$. This cannot have any eigenvalues, otherwise, there would be some λ such that $(0, x_1, x_2, x_3, \dots) = (\lambda x_1, \lambda x_2, \lambda x_3, \dots)$. We see that no matter what value λ assumes, it forces x_n to be zero for all $n \in \mathbb{N}$, and so we would have a zero eigenvector, a clear contradiction.

As the previous example shows, not all operators have eigenvalues. We therefore, want to generalise the concept of eigenvalue, which leads to the notion of *spectrum.*

Definition 4.5. Let $A \in B(H)$ where H is a complex Hilbert space. The *spectrum of* A is defined as

$$\sigma(A) := \{\lambda \in \mathbb{C} \mid A - \lambda I \text{ is not bijective } \}.$$

Similarly, the *resolvent set of* A is defined as

$$\rho(A) := \{\lambda \in \mathbb{C} \mid A - \lambda I \text{ is bijective and } (A - \lambda I)^{-1} \text{ is bounded } \}.$$

Remark 4.6. By the bounded inverse theorem A.23, if $A - \lambda I$ is bijective then $(A - \lambda I)^{-1}$ is bounded as a consequence. Thus $\sigma(A) = \mathbb{C} \setminus \rho(A)$.

Remark 4.7. The spectrum $\sigma(A)$ for any operator $A \in B(H)$ is always non-empty. For a proof of this, see [9], proposition 7.5.

Definition 4.8. Let $A \in B(H)$ where H is a complex Hilbert space. We then define the following sets.

- *Point Spectrum* :
  $\sigma_p(A) = \{\lambda \in \mathbb{C} \mid A - \lambda I \text{ is not injective } \}$.
- *Continuous Spectrum* :
  $\sigma_c(A) = \{\lambda \in \mathbb{C} \mid A - \lambda I \text{ injective but not surjective and } \overline{Ran(A - \lambda I)} = H\}$.
- *Resolvent Spectrum* :
  $\sigma_r(A) = \{\lambda \in \mathbb{C} \mid A - \lambda I \text{ injective but not surjective and } \overline{Ran(A - \lambda I)} \neq H\}$.

Proposition 4.9. *Let $A \in B(H)$. Then $\sigma(A) = \sigma_p(A) \cup \sigma_c(A) \cup \sigma_r(A)$.*

Proof: All that is needed to show is $\sigma(A) \subseteq \sigma_p(A) \cup \sigma_c(A) \cup \sigma_r(A)$, since the reverse inclusion is trivial.
Let $\lambda \in \sigma(A)$. Then $A - \lambda I$ is not bijective, so either it is not injective, or it is not surjective. If it is not injective, then $\lambda \in \sigma_p(A)$. If it is injective, then then it must not be surjective, i.e. $Ran(A - \lambda I) \neq H$. This decomposes further into the case where $Ran(A - \lambda I)$ is dense in H or not, giving the cases $\lambda \in \sigma_c(A)$ and $\lambda \in \sigma_r(A)$ respectively. ■

Example 4.10. Let $T : \mathbb{C}^n \to \mathbb{C}^n$ be a linear map with corresponding matrix $\mathbf{T} := [T_{ij}]$. From basic linear algebra, we know that T has n eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ when counting for multiplicity and that the eigenvalues are the roots of the characteristic polynomial $P(\lambda) = \det(\mathbf{T} - \lambda \mathrm{id})$. Since $(T - \lambda \mathrm{id})^{-1}$ is bounded when λ is not an eigenvalue, it follows that the spectrum of T is comprised entirely of the point spectrum. Thus the resolvent set is the complex plane less a finite number of points, namely $\rho(T) = \mathbb{C} \setminus \{(\lambda_k)_{k=1}^n\}$.

Lemma 4.11 (Neumann's Lemma)**.** *Let H be a Hilbert space and let $T \in B(H)$ be an operator with norm less than* 1*. Then the operator $I - T$ has an inverse operator $(I - T)^{-1}$ on H.*

Proof: Consider the infinite series $S = I + \sum_{n=1}^{\infty} T^n = I + T + T^2 + \cdots + T^n + \ldots$. Using the fact that absolute convergence implies convergence in a Hilbert space A.21, the S converges by the comparison test because $\|T^n\| \leq \|T\|^n$ for all $n \in \mathbb{N}$ and the series $\sum_{n=0}^{\infty} \|T\|^n$ is convergent since $\|T\| < 1$. We then make the following calculations.

$$(I - T)\left(\sum_{k=0}^{n} T^k\right) = \left(\sum_{k=0}^{n} T^k\right)(I - T) = \sum_{k=0}^{n}(T^k - T^{k+1}) = I - T^{n+1}.$$

Since $\|T^n - \mathbf{0}\| = \|T^n\| \leq \|T\|^n \to 0$, it follows that

$$\lim_{n\to\infty}(I - T)\left(\sum_{k=0}^{n} T^k\right) = I = \lim_{n\to\infty}\left(\sum_{k=0}^{n} T^k\right)(I - T),$$

or equivalently,

$$(I - T)S = (I - T).$$

Thus, $(I - T)$ is invertible with inverse operator equal to S. ■

Theorem 4.12. *Let H be a Hilbert space and let $A \in B(H)$ be an operator on H. Then the resolvent set $\rho(A)$ is open, hence, the spectrum $\sigma(A)$ is closed.*

Proof: If $\rho(A) = \emptyset$ then $\rho(A)$ is obviously closed, so suppose $\rho(A) \neq \emptyset$. Let $\mu \in \rho(A)$. Then $(A - \mu I)^{-1}$ is an operator on H. For $\lambda \in \mathbb{C}$ we have the following,

$$A - \lambda I = (A - \mu I) - (\lambda - \mu)I = (A - \mu I)(I - (\lambda - \mu)(A - \mu I)^{-1}).$$

By Neumann's Lemma 4.11, this shows that $A - \lambda I$ is an invertible operator with bounded inverse, whenever we have that

$$\|(\lambda - \mu)(A - \mu I)^{-1}\| = |\lambda - \mu|\|(A - \mu I)^{-1}\| < 1,$$

or equivalently, whenever $|\lambda - \mu| < \frac{1}{\|A - \mu I\|^{-1}}$. This inequality describes a circle in $\mathbb{C}$ with centre μ, proving that $\rho(A)$ is open.
Lastly, the closedness of the spectrum follows from the fact that $\sigma(A) = \mathbb{C}\backslash\rho(A)$. ■

Theorem 4.13. *Let H be a Hilbert space and let $A \in B(H)$ be an operator on H. Then the spectrum $\sigma(A)$ is a compact set in $\mathbb{C}$ contained in the circle of radius $\|A\|$ centred about the origin.*

Proof: Let $\lambda \in \mathbb{C}$ be such that $|\lambda| > \|A\|$. We then have that

$$A - \lambda I = -\lambda \left(I - \frac{1}{\lambda} A \right)$$

has a bounded inverse since $\| \left(\frac{1}{\lambda}\right) A\| < 1$ by Neumann's Lemma 4.11, hence $\lambda \in \rho(A)$. Consequently, $\lambda \in \sigma(A)$ implies that $|\lambda| \leq \|A\|$, and so $\sigma(A)$ is bounded. However, from 4.12 we know that $\sigma(A)$ is also closed. By the Bolzano–Weierstrass theorem, it follows $\sigma(A)$ is compact. ■

4.2 The Spectral Theorem for Compact Self-Adjoint Operators.

In this section, we state and prove the spectral theorem for compact self-adjoint operators. A guiding source is section 5.3 in [10].

For a compact self-adjoint operator A on a Hilbert space H, by 3.8, we have the following alternative formula for calculating the operator norm $\|A\|$, namely

$$\|A\| = \sup\{|(Ax \mid x)| \mid x \in H_1\}.$$

Proposition 4.14. *Let $A \in B(H)$ be a self-adjoint operator on a Hilbert space H. Then for all λ in the spectrum of A it holds that*

$$|\lambda| \leq \sup\{|(Ax \mid x)| \mid x \in H_1\}.$$

Proof: This follows as a direct consequence of theorem 4.13. ■

Example 4.15. Let H be an infinite dimensional separable Hilbert space. By theorem 2.36, H has a countable orthonormal basis, call it $(e_n)_{n\in\mathbb{N}}$. We define a map $A : H \to H$ called the *left shift* on H with respect to this basis, defined by the following,

$$A(x) = A\left(\sum_{n=1}^{\infty}(x \mid e_n)e_n\right) := \sum_{n=2}^{\infty}(x \mid e_n)e_{n-1}, \text{ for all } x \in H.$$

It can be shown that A is bounded and linear. We determine the spectrum $\sigma(A)$. Suppose $x \in H$ is an eigenvector for A. Then there exists $\lambda \in \mathbb{C}$ such that

$$A(x) = A\left(\sum_{n=1}^{\infty}(x \mid e_n)e_n\right) = \sum_{n=2}^{\infty}(x \mid e_n)e_{n-1} = \lambda\sum_{n=1}^{\infty}(x \mid e_n)e_n.$$

This gives a recursive relation and for each $n \in \mathbb{N}_{\geq 2}$ we have $(x \mid e_n) = \lambda^{n-1}(x \mid e_{n-1})$.

Suppose $(x \mid e_1) = 1$. Then $(x \mid e_n) = \lambda^{n-1}$ for all $n \in \mathbb{N}_{\geq 2}$, and hence $x = \sum_{n=1}^{\infty} \lambda^{n-1} e_n$ is an eigenvector corresponding to λ if and only if the sum defines a vector $x \in H$, and since H is complete, it is easy to see that this happens if and only if $\sum_{n=1}^{\infty} |\lambda^{n-1}|^2 < \infty$. The latter is the case if and only if $|\lambda| < 1$. Hence, the open unit disc is contained in the spectrum $\sigma(A)$.

Conversly, for all $x \in H$ we have that

$$\|Ax\|^2 = \sum_{n=2}^{\infty} |(x \mid e_n)|^2 \leq \sum_{n=1}^{\infty} |(x \mid e_n)|^2 = \|x\|^2.$$

Since $\|Ae_n\| = 1$ for all $n \geq 2$ we also get that $\|A\| = 1$. By theorem 4.13, this implies that $\sigma(A)$ is contained in the closed unit disc about 0. Moreover, since $\sigma(A)$ is closed by 4.12, it follows that $\sigma(A)$ is exactly the closed unit disc in $\mathbb{C}$.

Theorem 4.16. *Let $A : H \to H$ be a self-adjoint operator on a Hilbert space H. Then all of its eigenvalues belong to $\mathbb{R}$. Moreover, any pair of eigenvectors corresponding to different eigenvalues is orthogonal.*

Proof: By the self-adjoint property of A, we have that

$$(Ax \mid y) = (x \mid Ay), \text{ for all } x, y \in H.$$

If λ is an eigenvalue with corresponding non-zero eigenvector $x \in H$ we then have

$$\lambda(x \mid x) = (\lambda x \mid x) = (Ax \mid x) = (x \mid Ax) = (x \mid \lambda x) = \overline{\lambda}(x \mid x).$$

Since x is not zero, $(x \mid x)$ is non-zero also, hence $\lambda = \overline{\lambda}$, proving the first claim.
For the second bit, suppose we have another distinct eigenvalue $\mu \in \mathbb{C}$ with some corresponding eigenvector $0 \neq y \in H$. We then have the following.

$$\lambda(x \mid y) = (\lambda x \mid y) = (Ax \mid y) = (x \mid Ay) = (x \mid \mu y) = \overline{\mu}(x \mid y) = \mu(x \mid y).$$

Since λ and μ are distinct, we have that $(x \mid y) = 0$, hence $x \perp y$. ■

Definition 4.17. Let $A \in B(H)$ be an operator on a Hilbert space H. We say a scalar $\lambda \in \mathbb{C}$ is an *approximate eigenvalue* for A, if there exists a sequence of unit vectors $(x_n)_{n\in\mathbb{N}} \in H^{\mathbb{N}}$ such that

$$\lim_{n\to\infty} (A - \lambda I)x_n = 0.$$

The set of approximate eigenvalues is called the *approximate point spectrum* and is denoted by $\sigma_{ap}(A)$.

Theorem 4.18. *Let $A \in B(H)$ be a self-adjoint operator on a Hilbert space H. Then $\sigma_{ap}(A)$ is non-empty.*

Proof: Recall that since $\|A\| = \sup\{|(Ax \mid x)| \mid x \in H_1\}$, there exists a sequence of unit vectors $(y_n)_{n\in\mathbb{N}}$ in H such that $\lim_{n\to\infty} |(Ay_n \mid y_n)| = \|A\|$.
Since A is self-adjoint, $(Ay_n \mid y_n)$ is a real number, hence there is a subsequence $(x_n)_{n\in\mathbb{N}}$ of $(y_n)_{n\in\mathbb{N}}$ such that either $\lim_{n\to\infty}(Ax_n \mid x_n) = \|A\|$, or $\lim_{n\to\infty}(Ax_n \mid x_n) = -\|A\|$. Let λ be either positive or negative $\|A\|$. We then have the following,

$$\begin{aligned}
\|(A - \lambda I)x_n\|^2 &= (Ax_n - \lambda x_n \mid Ax_n - \lambda x_n)\\
&= \|Ax_n\|^2 + \lambda^2\|x_n\|^2 - 2\lambda(Ax_n \mid x_n)\\
&\leq \|A\|^2 + \lambda^2 - 2\lambda(Ax_n \mid x_n)\\
&= 2\lambda^2 - 2\lambda(Ax_n \mid x_n) \to 2\lambda^2 - 2\lambda^2 = 0. \,(\text{As}\, n \to \infty)
\end{aligned}$$

Thus, $\|(A - \lambda I)x_n\| \to 0$, proving the claim. ■

Remark 4.19. For an operator $A \in B(H)$, the approximate point spectrum is contained in the spectrum, i.e. $\sigma_{ap}(A) \subseteq \sigma(A)$. To see this, suppose $\lambda \in \sigma_{ap}(A)$, so there is a sequence of unit vectors $(x_n)_{n\in\mathbb{N}} \in H^{\mathbb{N}}$ such that $\|(A-\lambda I)x_n\| \to 0$. Thus $A-\lambda I$ is clearly not bounded from below, that is, there is no constant $C > 0$ such that $\|(A - \lambda I)x\| \leq C\|x\|$ for all $x \in H$. Since an operator is not invertible if it is not bounded below A.24, it follows that $A - \lambda I$ is not bijective, hence $\lambda \in \sigma(A)$.

Our next result shows that a compact, self-adjoint operator on a Hilbert space always has an eigenvalue.

Theorem 4.20. *Let $A \in B(H)$ be a compact, self-adjoint operator on a Hilbert space H. Then at least one of $\|A\|$ or $-\|A\|$ is an eigenvalue for A.*

Proof: If $\|A\| = 0$ the claim follows trivially, so suppose $\|A\| \neq 0$.
Let $\lambda = \pm\|A\|$. By theorem 4.18 there is a sequence of unit vectors $(x_n)_{n\in\mathbb{N}}$ in H such that $\lim_{n\to\infty}(A - \lambda I)x_n = 0$, where λ is either $\|A\|$ or $-\|A\|$. Since A is compact, the sequence $(x_n)_{n\in\mathbb{N}}$ has a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $(Ax_{n_k})_{k\in\mathbb{N}}$ is convergent. Since $\lim_{k\to\infty}(Ax_{n_k} - \lambda x_{n_k}) = 0$ it follows that $\lim_{k\to\infty} \lambda x_{n_k} = \lim_{k\to\infty}[Ax_{n_k} - (Ax_{n_k} - \lambda x_{n_k})]$ exists. Moreover, multiplication by a scalar does not affect convergence, thus $\lim_{k\to\infty} x_{n_k} = x$ for some $x \in H$. By continuity of the norm A.30, since $\|x_n\| = 1$ for all $n \in \mathbb{N}$, we also have $\|x\| = 1$.
By construction, we have $Ax = \lambda x$, where $\lambda = \pm\|A\|$ and $x \neq 0$, proving the claim.
■

Corollary 4.20.1. *Let A be a compact, self-adjoint operator on a Hilbert space H. Then the operator norm is given by*

$$\|A\| = \max\{|(Ax \mid x)| \mid x \in H_1\}.$$

Moreover, the maximum value is attained for a non-zero eigenvector corresponding to the eigenvalue $\|A\|$ or $-\|A\|$.

Proof: By theorem 4.20, there is a unit vector $x \in H$ such that $Ax = \lambda x$, where $\lambda = \pm\|A\|$. From this it follows that

$$\|Ax\| = \|\lambda x\| = |\lambda|\|x\| = \|A\| = \sup\{|(Ax \mid x)| \mid x \in H_1\}.$$

Thus, the supremum is attained at x and can be replaced with a maximum. ■

Proposition 4.21. *Let A be a compact operator on a Hilbert space H, and suppose that A has a non-zero eigenvalue λ. Then the eigenspace associated to λ, Eig(λ), is finite dimensional.*

Proof: Suppose otherwise, that is, Eig(λ) is infinite-dimensional. Choose an orthonormal sequence $(x_n)_{n\in\mathbb{N}}$ in Eig(λ) such that

$$\|Ax_n - Ax_m\|^2 = |\lambda|^2\|x_n - x_m\|^2 = 2|\lambda|^2 > 0, \text{ for } n \neq m.$$

Thus $(Ax_n)_{n\in\mathbb{N}}$ has no convergent subsequences, hence A cannot be compact. This provides the necessary contradiction. ■

We are now ready to prove the main result of this section.

Theorem 4.22 (Spectral theorem - Compact Operators.)**.** *Let A be a compact, self-adjoint operator on a separable Hilbert space H of infinite dimension. Then H admits an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ consisting of eigenvectors of A. Moreover, the enumeration of the infinite sequence of basis vectors $(e_n)_{n\in\mathbb{N}}$ can be chosen such that the sequence of corresponding eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$ decreases numerically,*

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n| \geq \ldots, \text{ and } \lambda_n \to 0 \text{ as } n \to \infty.$$

Lastly, we may describe A using the basis of eigenvectors as follows. For $x \in H$, we have

$$Ax = \sum_{n=1}^{\infty} \lambda_n (x \mid e_n)e_n, \text{ where } x = \sum_{n=1}^{\infty}(x \mid e_n)e_n.$$

Proof: By corollary 4.20.1, the compact, self-adjoint operator A has at least one eigenvalue, $\lambda_1 = \pm\max\{|(Ax \mid x)| \mid x \in H_1\}$. Let $e_1 \in H$ be a corresponding unit eigenvector.
Let $V_1 = \text{span}\{e_1\}^\perp$ be the orthogonal complement of the $1-$dimensional subspace spanned by the vector e_1. By 2.22, V_1 is a closed subspace of H and hence is also a Hilbert space. For $x \in V_1$, since $\lambda_1 \in \mathbb{R}$, we have the following,

$$(Ax \mid e_1) = (x \mid Ae_1) = \lambda_1(x \mid e_1) = 0.$$

Thus if $x \in V_1$, $Ax \in V_1$, and so V_1 is invariant under A. Thus A can be considered as a compact, self-adjoint operator on the Hilbert space V_1. Using theorem 4.20 again, we get a second eigenvalue for A within V_1,

$$\lambda_2 = \pm \max\{|(Ax \mid x)| \mid x \in V_1, \|x\| \leq 1\},$$

and a corresponding unit vector $e_2 \in V_1$ for λ_2. It is clear from construction that $|\lambda_1| \geq |\lambda_2|$ and that $e_1 \perp e_2$.
Next, we repeat this process and let $V_2 = \operatorname{span}\{e_1, e_2\}^{\perp}$. Clearly $(Ax \mid e_1) = 0$ and $(Ax \mid e_2) = 0$ where $x \in \operatorname{span}\{e_1, e_2\}^{\perp}$, and so V_2 is invariant under A, which can thus be considered a compact self-adjoint operator on V_2. Once again, theorem 4.20 provides a third eigenvalue λ_3 corresponding to a unit eigenvector e_3 also in V_2. Continuing on in this manner, we get an orthonormal sequence of eigenvectors $(e_n)_{n \in \mathbb{N}}$ and a decreasing sequence $(\lambda_n)_{n \in \mathbb{N}}$ of corresponding eigenvalues associated with the following sequence of subspaces,

$$\cdots \subseteq V_n \subseteq V_{n-1} \subseteq \cdots \subseteq V_1 \subseteq H.$$

From example 2.48, $(e_n)_{n \in \mathbb{N}}$ converges weakly to 0, and since A is compact, by 3.23, $(Ae_n)_{n \in \mathbb{N}}$ converges to 0 strongly. This proves that $\lim_{n\to\infty} |\lambda_n| = \lim_{n\to\infty} \|Ae_n\| = 0$.

Next, let $U = \left\{x \in H \mid x = \sum_{n=1}^{\infty} \alpha_n e_n, \text{ for some } (\alpha_n)_{n \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}}\right\}$. Then $(e_n)_{n \in \mathbb{N}}$ is an orthonormal basis for U.
To see this, suppose $F \subseteq U$ is another orthonormal system such that $\{e_n \mid n \in \mathbb{N}\} \subseteq F$, and let $f \in F$. By definition, there is $(\alpha_n)_{n \in \mathbb{N}}$ such that $f = \sum_{n=1}^{\infty} \alpha_n e_n$. Let $e_m \in \{e_n \mid n \in \mathbb{N}\}$ be arbitrary and consider the following.

$$\begin{aligned}(f \mid e_m) &= \sum_{n=1}^{\infty} \alpha_n (e_n \mid e_m) \\ &= 0 + 0 + \dots \alpha_m (e_m \mid e_m) + 0 + \cdots = \alpha_m.\end{aligned}$$

Thus, if $(f \mid e_m)$ were 0 for every $m \in \mathbb{N}$, then $\alpha_m = 0$ for every $m \in \mathbb{N}$ also. Then f would be the zero vector, contradicting its membership of F. Thus there exists some $m \in \mathbb{N}$ such that $(f \mid e_m) = 1$ and since F is orthonormal, it follows $f = e_m$, and so $F \subseteq \{e_n \mid n \in \mathbb{N}\}$.
Moreover, each $x \in U$ has a unique decomposition with respect to $(e_n)_{n \in \mathbb{N}}$. To see this, suppose $x = \sum_{n=1}^{\infty} a_n e_n = \sum_{n=1}^{\infty} \beta_n e_n$. For a fixed $m \in \mathbb{N}$, we have

$$
\begin{aligned}
0 &= (x - x \mid e_m) \\
&= \left(\sum_{n=1}^{\infty} (\alpha_n - \beta_n) e_n \mid e_m \right) \\
&= \sum_{n=1}^{\infty} \alpha_n (e_n \mid e_m) - \sum_{n=1}^{\infty} \beta_n (e_n \mid e_m) \\
&= \alpha_m (e_m \mid e_m) - \beta_m (e_m \mid e_m) = \alpha_m - \beta_m.
\end{aligned}
$$

Thus, $\alpha_n = \beta_n$ for all natural numbers n and so the decomposition is unique.

This allows us to define a natural isomorphism $T : U \to \ell^2$ that maps a basis element e_n in U to the the basis vector $\underbrace{(0, 0, \ldots, 1, 0 \ldots,)}_{\text{In the } n^{th} \text{ slot}}$ in ℓ^2. For $x = (x_n)_{n \in \mathbb{N}} \in \ell^2$, we have that

$$
T \left(\sum_{n=1}^{\infty} x_n e_n \right) = x,
$$

and so T is surjective. Furthermore, for any $x = \sum_{n=1}^{\infty} \alpha_n e_n \in U$ we have

$$
\|Tx\| = \sqrt{\sum_{n=1}^{\infty} \alpha_n^2} = \sqrt{(x \mid x)} = 1 \cdot \|x\|.
$$

Thus, $U \cong \ell^2$, and since Isometric isomorphisms preserve completeness, it follows that U is closed.

We next show that A is the zero operator when restricted to $U^{\perp}$, i.e. $Ay = 0$, for all $y \in U^{\perp}$.
This is trivially true for $y = 0$ so suppose $y \in U^{\perp}$ is non-zero. Let $y_1 = \frac{y}{\|y\|}$, and so y can be written as $y = \|y\| y_1$. We then get that $(Ay \mid y) = \|y\|^2 (Ay_1 \mid y_1)$. Since $y_1 \in V_n$ for all $n \in \mathbb{N}$, it follows that $|(Ay_1 \mid y_1)| \leq |\lambda_n|$ for all $n \in \mathbb{N}$, and consequently that

$$
|(Ay \mid y)| \leq \|y^2\| |\lambda_n| \to 0 \text{ as } n \to \infty.
$$

We conclude that $(Ay \mid y) = 0$ for all $y \in U^{\perp}$, hence A must be the zero operator on $U^{\perp}$.

Now let $(f_n)_{n \in \mathbb{N}} \in \left(U^{\perp}\right)^{\mathbb{N}}$ be an orthonormal basis for $U^{\perp}$, so that $\{e_n \mid n \in \mathbb{N}\} \cup \{f_n \mid n \in \mathbb{N}\}$ is an orthonormal basis for H. Since $Af_n = 0$ for all $n \in \mathbb{N}$, each f_n is is an eigenvector for A with corresponding eigenvalue equal to 0. For each $n \in \mathbb{N}$, let $(a_n)_{n\in\mathbb{N}}$ be the sequence given as follows,

$$a_n = \begin{cases} e_k & n \,\text{is odd and}\, n = 2k-1 \\ f_k & n \,\text{is even and}\, n = 2k \end{cases}$$

This gives $(a_n)_{n\in\mathbb{N}} = (e_1, f_1, e_2, f_2, \dots)$ with corresponding sequence of eigenvalues $(\lambda_1, 0, \lambda_2, 0, \lambda_3, 0, \dots)$. Cleary this sequence of eigenvalues still converges to 0 since all the new eigenvalues added to the sequence are 0.

Finally, from 2.30 for each $x \in H$ we have $x = \sum_{n=1}^{\infty}(x \mid a_n)a_n$. By continuity of A we then have the following,

$$\begin{aligned} Ax &= A\left(\sum_{n=1}^{\infty}(x \mid a_n)a_n\right) \\ &= \sum_{n=1}^{\infty}(x \mid a_n)Aa_n \\ &= \sum_{n=1}^{\infty}\lambda_n(x \mid e_n)e_n. \end{aligned}$$

Thus $(a_n)_{n\in\mathbb{N}}$ satisfies the criteria of the theorem and the proof is complete. ■

4.3 Projection Valued Measures and the General Spectral Theorem.

In the last section of this chapter, we provide a version of the spectral theorem for self-adjoint operators, relaxing the added assumption of compactness. This requires moving away from the discrete setting of an infinite sum seen in the last section. Although the modern approach in the literature uses techniques from operator theory and C^* algebras, here we manage to avoid this and provide a proof using solely measure-theoretic arguments. Guiding sources for this section include section 2.5 in [13], chapter 9, section 2 in [6], and chapters 7 and 8 in [9]. General concepts in measure theory are also used from [2].

Recall from section 2.3, that for a closed subspace S of a Hilbert space H, there is a unique decomposition of H as follows, $H = S \oplus S^\perp$. As a result, we have a unique map $P_S : H \to H$ called the *orthogonal projection* onto S, some properties of which are listed in remark 2.27. Another important property of an P_S is that it is always self-adjoint. To see this, suppose x_1, x_2 are elements in H with unique decompositions $y_1 + z_1$ and $y_2 + z_2$ respectively, both in $S \oplus S^\perp$. We then have the following,

$$\begin{aligned}(P_S(x_1) \mid x_2) &= (y_1 \mid y_2 + z_2)\\ &= (y_1 \mid y_2) + (y_1 \mid z_2)\\ &= (y_1 \mid y_2) + (z_1 \mid y_2)\\ &= (x_1 \mid y_2) = (x_1 \mid P_S(x_2))\end{aligned}$$

It turns out that it is convenient to describe closed subspaces of a Hilbert space in terms of the associated orthogonal projection operators, when one wants to formulate a spectral theorem that doesn't require compactness. The function that associates these subspaces to orthogonal projections has properties similar to those of a measure, and so the term *projection valued measure* is used.

Definition 4.23. Let X be a set and let Ω be a σ-algebra on X. A function $\mu : \Omega \to B(H)$ is called a *projection valued measure* on X if the following conditions are satisfied.

(i) For each $E \in \Omega$, $\mu(E)$ is an orthogonal projection.

(ii) $\mu(\emptyset) = \mathbf{0}$ and $\mu(X) = \mathrm{id}_H$.

(iii) If $(E_n)_{n\in\mathbb{N}} \in \Omega^{\mathbb{N}}$ is a sequence of mutually disjoint elements of the σ-algebra Ω, then for all $x \in H$ we have that

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) x = \sum_{n=1}^{\infty} \mu(E_n)x.$$

(iv) For every $E_1, E_2 \in \Omega, \mu(E_1 \cap E_2) = \mu(E_1)\mu(E_2)$.

Remark 4.24. For any projection valued measure μ and any element x in the Hilbert space, we can form an ordinary real-valued measure μ_x by the following relation.

$$\mu_x(E) := (\mu(E)x \mid x), \text{ where } E \in \Omega.$$

This observation allows us to construct a link between integration with respect to a projection-valued measure and integration with respect to an ordinary measure.

Our first result of this section shows that the spectrum of $p(A)$, where p is some polynomial and A is an operator, consists exactly of numbers of the form $p(\lambda)$ where λ is in the spectrum of A.

Lemma 4.25 (Spectral Mapping Lemma)**.** *Let $A \in B(H)$ be an operator on a Hilbert space H and let p be a polynomial. Then,*

$$\sigma(p(A)) = p(\sigma(A))$$

Proof: First suppose that $\deg(p) = 0$, and so $p(x) = a$ for some constant $a \in \mathbb{C}$. Then $p(A) = a\,\mathrm{id}_H$. The map $x \mapsto (a-\lambda)x$ on H is always bijective unless $\lambda = a$, in which case it is not bijective. Thus $\sigma(p(A)) = \{a\}$. Conversely, $p(\sigma(A)) \subseteq p(\mathbb{C}) = \{a\}$, and so the claim holds for constant polynomials.

Now suppose $p(x) = \sum_{i=0}^{n} a_i x^i$ where $n \geq 1$ and let $\lambda \in \sigma(A)$. We then have that

$$\begin{aligned} p(A) - p(\lambda)\,\mathrm{id}_H &= a_n(A^n - \lambda^n\,\mathrm{id}_H) + a_{n-1}(A^{n-1} - \lambda^{n-1}\,\mathrm{id}_H) + \cdots + a_0(\mathrm{id}_H - \lambda\,\mathrm{id}_H) \\ &= \sum_{k=0}^{n} a_{n-k}(A^{n-k} - \lambda^{n-k}\,\mathrm{id}_H). \end{aligned}$$

Also note that for any k we can factor $A^k - \lambda^k \mathrm{id}_H$ in the following way,

$$A^k - \lambda^k \mathrm{id}_H = (A - \lambda\,\mathrm{id}_H)(A^{k-1} + \lambda\, A^{k-2} + \cdots + \lambda^{k-1}\,\mathrm{id}_H) = (A - \lambda\,\mathrm{id}_H)\sum_{i=1}^{k} \lambda^{i-1} A^{k-i}.$$

Combining these two gives the following,

$$\begin{aligned} p(A) - p(\lambda)\mathrm{id}_H &= (A - \lambda\,\mathrm{id}_H)\left(\sum_{k=0}^{n}\left[a_{n-k}\left(\sum_{i=1}^{n-k}\lambda^{i-1}A^{n-k-i}\right)\right]\right) \\ &:= (A - \lambda\,\mathrm{id}_H)q(A), \end{aligned}$$

where we have shortened the right-hand side to include q which is a polynomial depending on λ. Since $\lambda \in \sigma(A)$, $A - \lambda\mathrm{id}_H$ is not bijective, and it therefore follows that $(A - \lambda\,\mathrm{id}_H)q(A) = p(A) - p(\lambda)\,\mathrm{id}_H$ is also not bijective, hence $p(\lambda) \in \sigma(p(A))$. This gives the first inclusion, i.e. $p(\sigma(A)) \subseteq \sigma(p(A))$.

For the reverse, suppose $\alpha \in \sigma(p(A))$, that is, $p(A) - \alpha\,\mathrm{id}_H$ is not bijective. Since $p(z) - \alpha$ is a degree n polynomial in $\mathbb{C}$ which is algebraically closed, we can factor it by its roots $b_1, b_2, \ldots, b_n$ as follows,

$$p(z) - \alpha = c(z - b_1)(z - b_2)\ldots(z - b_n). \tag{4.1}$$

This in turn implies that

$$p(A) - \alpha\,\mathrm{id}_H = c(A - b_1\,\mathrm{id}_H)(A - b_2\,\mathrm{id}_H)\ldots(A - b_n\,\mathrm{id}_H).$$

Now since $P(A) - \alpha\,\mathrm{id}_H$ is not bijective, it follows there is $i \in \{1, \ldots, n\}$ such that $A - b_i\,\mathrm{id}_H$ is not bijective, thus $b_i \in \sigma(A)$. From 4.1 it follows that $p(b_i) - \alpha = 0$, hence $\alpha \in p(\sigma(A))$, completing the reverse inclusion.

■

Definition 4.26. Let $C(A, \mathbb{R})$ denote the set of real-valued continuous functions defined on a set $A \subseteq \mathbb{C}$, that is,

$$C(A, \mathbb{R}) = \{f \in \mathbb{R}^A \mid f \text{ is continuous}\}.$$

Definition 4.27. Let $A \in B(H)$ be an operator. The *spectral radius of* A, denoted by $r(A)$ is the largest value in magnitude of the spectral values of A, that is,

$$r(A) = \sup_{\lambda \in \sigma(A)} |\lambda|.$$

Remark 4.28. By theorem 4.13, for any operator $A \in B(H)$, we have $r(A) \leq \|A\|$. Furthermore, for self-adjoint operators, theorem 4.18 says that both $\|A\|$ and $-\|A\|$ are approximate eigenvalues for A. Since $\sigma_{ap}(A) \subseteq \sigma(A)$ by remark 4.19, it follows $\|A\| \in \sigma(A)$, hence the spectral radius and the norm coincide for the class of self-adjoint operators.

Our next main results show in fact that spectrum is contained entirely in the real line for self-adjoint operators. We first prove a useful lemma.

Lemma 4.29. *Suppose $A \in B(H)$ is self-adjoint. Then for all $\lambda = a + ib \in \mathbb{C}$ we have*

$$((A - \lambda \operatorname{id}_H)x \mid (A - \lambda \operatorname{id}_H)x) \geq b^2(x \mid x).$$

Proof: Let A and λ be given as in the hypotheses and compute the following.

$$\begin{aligned}((A - (a+ib)\operatorname{id}_H)x \mid (A - (a+ib)\operatorname{id}_H)x) &= ((A - a\operatorname{id}_H)x \mid (A - a\operatorname{id}_H)x) \\ + ib((A - a\operatorname{id}_H)x \mid x) - ib(x \mid (A - a\operatorname{id}_H)x) + b^2(x \mid x).\end{aligned}$$

Since A is self-adjoint, we see that $(A - a\operatorname{id}_H)^* = A^* - a\operatorname{id}_H{}^* = A - a\operatorname{id}_H$ and so $A - a\operatorname{id}_H$ is self-adjoint also. It follows that the second and third terms on the right-hand side of the above equation cancel, leaving the following,

$$((A - (a+ib)\operatorname{id}_H)x \mid (A - (a+ib)\operatorname{id}_H)x) = ((A - a\operatorname{id}_H)x \mid (A - a\operatorname{id}_H)x) + b^2(x \mid x),$$

from which the result follows. ■

Theorem 4.30. *Let $A \in B(H)$ be an operator on a Hilbert space H. If A is self-adjoint, then $\sigma(A) \subseteq \mathbb{R}$.*

Proof: Let $A \in B(H)$ be self-adjoint. Suppose $\lambda = a + ib$ and b is non-zero. We wish to show $\lambda \notin \sigma(A)$, or equivalently by remark 4.6, that $\lambda \in \rho(A)$. To this end we must show that $A - \lambda\operatorname{id}_H$ is bijective and $(A - \lambda\operatorname{id}_H)^{-1}$ is bounded. By 3.4 we have that $\operatorname{Ran}(A - \lambda\operatorname{id}_H)^{\perp} = \ker((A - \lambda\operatorname{id}_H)^*) = \ker(A - \overline{\lambda}\operatorname{id}_H)$. Thus by orthogonal decomposition 2.25 we have that

$$H = \overline{\operatorname{Ran}(A - \lambda\operatorname{id}_H)} \oplus \ker(A - \overline{\lambda}\operatorname{id}_H) \tag{4.2}$$

Suppose $x \in \ker(A - \overline{\lambda}\operatorname{id}_H)$. Using the fact that A is self-adjoint, we get the following set of equations,

$$\begin{aligned}\lambda(x \mid x) = (x \mid \overline{\lambda}x) &= (x \mid Ax) \\ &= (Ax \mid x) \\ &= (\overline{\lambda}x \mid x) \\ &= \overline{\lambda}(x \mid x).\end{aligned}$$

Since $\lambda \neq \overline{\lambda}$, we must have $x = 0$, and so $\ker(A - \overline{\lambda}\operatorname{id}_H) = \{0\}$. Thus by (4.2), $\operatorname{Ran}(A - \lambda\operatorname{id}_H)$ is dense in H. To show it is in fact the whole space H, let $y \in H$

and suppose $y_n = (A - \lambda\,\mathrm{id}_H)x_n$ is a sequence in $\mathrm{Ran}(A - \lambda\,\mathrm{id}_H)$ converging to y. Since $(y_n)_{n\in\mathbb{N}}$ is convergent, it is a Cauchy sequence, hence for $n, m \in \mathbb{N}$ we have the following,

$$\begin{aligned}
0 \leftarrow \|y_n - y_m\| &= \|(A - \lambda\,\mathrm{id}_H)x_n - (A - \lambda\,\mathrm{id}_H)x_m\| \\
&= \|(A - \lambda\,\mathrm{id}_H)(x_n - x_m)\| \\
&\geq b^2\|x_n - x_m\|, \text{ by lemma 4.29.}
\end{aligned}$$

Thus $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and hence by completeness of H, $\lim_{n\to\infty} x_n := x$ exists and is in H. By boundedness of A, and hence continuity A.15, we have that,

$$(A - \lambda\,\mathrm{id}_H)x = \lim_{n\to\infty}(A - \lambda\,\mathrm{id}_H)x_n = \lim_{n\to\infty} y_n = y.$$

It follows that $A - \lambda\,\mathrm{id}_H$ is surjective. Moreover, a very similar application of the inequality from 4.29 as used above shows it is also injective, hence bijective. By the bounded inverse theorem A.23, it follows that $(A - \lambda \mathrm{id}_H)^{-1}$ is bounded. ∎

Theorem 4.31 (Functional Calculus)**.** *Let $A \in B(H)$ be a self-adjoint operator on a Hilbert space H. Then there exists a unique bounded linear mapping from $C(\sigma(A), \mathbb{R})$ into $B(H)$, denoted by $f \mapsto f(A)$, such that when $f(x) = x^n$, we have $f(A) = A^n$. The map $f \mapsto f(A)$ is called the real-valued functional calculus on A.*

Proof: Suppose $A \in B(H)$ is self-adjoint and let $p(x) = \sum_{i=0}^n a_i x^i$ be a real-valued polynomial. We then have that

$$\begin{aligned}
p(A)^* &= \left(\sum_{i=0}^n a_i A^i\right)^* \\
&= \sum_{i=0}^n \overline{a_i}(A^*)^i \\
&= \sum_{i=0}^n a_i A^i = p(A),
\end{aligned}$$

where we have used the properties of the adjoint as listed in section 3.1. This shows that $p(A)$ is self-adjoint, and thus from the previous remark and the spectral mapping lemma 4.25 we have that

$$\begin{aligned}\|p(A)\| = r(p(A)) &= \sup_{\alpha \in \sigma(p(A))} |\alpha| \\ &= \sup_{\alpha \in p(\sigma(A))} |\alpha| \\ &= \sup_{\lambda \in \sigma(A)} |p(\lambda)|.\end{aligned}$$

This shows that the map $p \mapsto p(A)$ is an isometry between the collection of real-valued polynomials on $\sigma(A)$ with the supremum norm into $B(H)$ with the usual operator norm, hence it is bounded. Moreover, it is also easily seen to be linear, as $\alpha p + q \mapsto \alpha p(A) + q(A)$, where q is another real-valued polynomial. By the Stone-Weierstrass theorem [19], page 475, polynomials are dense in $C(\sigma(A), \mathbb{R})$. By 2.52, it follows we can extend the map $p \mapsto p(A)$ uniquely to a bounded linear operator from $C(\sigma(A), \mathbb{R})$ to $B(H)$. That $f(A) = A^n$ whenever $f(x) = x^n$ is precisely to say that this map is an extension. ■

Definition 4.32. Let $A \in B(H)$ be an operator. We say that A is *non-negative*, if $(Ax \mid x) \geq 0$ for all $x \in H$. We sometimes denote this by $A \geq 0$.

Proposition 4.33. *Let A be a self-adjoint operator on a Hilbert space H. Then for all $f, g \in C(\sigma(A), \mathbb{R})$ the functional calculus on A has the following properties.*

(i) ***Multiplicative:***

$(fg)(A) = f(A)g(A)$, *where fg denotes the pointwise product of f and g.*

(ii) ***Self-Adjoint:*** *$f(A)$ is self-adjoint.*

(iii) ***Non-negative:*** *$f(A)$ is non-negative if f is non-negative.*

(iv) ***Norm and Spectrum:*** *The norm can be evaluated by the following,*

$$\|f(A)\| = \sup_{\lambda \in \sigma(A)} |f(\lambda)|. \tag{4.3}$$

$$\sigma(f(A)) = \{f(\lambda) \in \mathbb{R} \mid \lambda \in \sigma(A)\}. \tag{4.4}$$

Proof: Let A be given as in the proposition and suppose $f, g \in C(\sigma(A), \mathbb{R})$.

(i) Let $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ be sequences of real-valued polynomials converging uniformly to f and g respectively, given by the Stone-Weierstrass theorem. We then have for all $n \in \mathbb{N}$, $(p_nq_n)(A) = p_n(A)q_n(A)$. This therefore gives

$$(fg)(A) = \lim_{n\to\infty}(p_nq_n)(A) = \lim_{n\to\infty} p_n(A)q_n(A) = f(A)g(A).$$

(ii) As seen in the proof of theorem 4.31, if p is real valued then $p(A)$ is self-adjoint. Thus, if $(p_n)_{n\in\mathbb{N}}$ is a sequence of real-valued polynomials converging to f, we have $f(A) = \lim_{n\to\infty} p_n(A)$ is the limit of self-adjoint operators, hence also self-adjoint.

(iii) Suppose f is non-negative, meaning $f(x) \geq 0$ for all $x \in \sigma(A)$. Thus $f(x) = g(x)^2$ for some $g \in C(\sigma(A), \mathbb{R})$ and for all $x \in \sigma(A)$, where $g = \sqrt{f}$. Thus, $g(A)$ is self-adjoint by (ii), and for all $x \in H$ we have that

$$(f(A)x \mid x) = (g(A)^2x \mid x) = (g(A)x \mid g(A)x) \geq 0,$$

showing that $f(A)$ is a non-negative operator as claimed.

(iv) For 4.3, we have already established the result for real-valued polynomials in 4.25 and so the claim holds for continuous functions on $\sigma(A)$ by taking appropriate limits.

For 4.4, first suppose $\lambda \in \mathbb{R}$ is not in the range of f, i.e. there is no $x \in \sigma(A)$ such that $f(x) = \lambda$. It follows therefore that the function $g : \sigma(A) \to \mathbb{R}$ given by $g(x) = \frac{1}{f(x)-\lambda}$ is continuous, hence $g \in C(\sigma(A), \mathbb{R})$. Moreover, we see that $g(A)$ is the inverse of $f(A) - \lambda I$, from which it follows $\lambda \notin \sigma(f(A)))$.
For the reverse direction, suppose $\lambda = f(x)$ for some $x \in \sigma(A)$. Suppose $f(x)$ is not in $\sigma(f(A))$ and so $f(A) - xI$ is bijective. Choose a sequence of real-valued polynomials $(p_n)_{n\in\mathbb{N}} \in \sigma(A)^{\mathbb{N}}$ converging uniformly to f. By A.25, there exists $\varepsilon > 0$ such that if B is an operator and $\|(f(A) - \lambda I) - B\| < \varepsilon$, then B is invertible. In particular, $p_n(A) - p_n(x)I$ would have to be invertible for sufficiently large n, which would contradict the spectral mapping lemma 4.25.

■

Our next result is an important tool in the proof of the spectral theorem. It is also named after the Hungarian mathematician Frigyes Riesz, many of whose results are important throughout functional analysis, as seen earlier with theorem 2.45 which was proven in chapter 2. A proof of the following theorem is quite involved and can be found in section 2.14 in [16].

Theorem 4.34 (Riesz-Representation)**.** *Let X be a compact metric space and let $\Gamma : C(X, \mathbb{R}) \to \mathbb{R}$ be linear with the property that $\Gamma(f)$ is non-negative whenever $f(x) \geq 0$ for all $x \in X$. Then there is a unique measure μ on the Borel $\sigma-$algebra in X for which*

$$\Gamma(f) = \int_X f\, d\mu, \textit{ for all} f \in C(X, \mathbb{R}).$$

Example 4.35. Let $A \in B(H)$ be self adjoint, and let $x \in H$. Define a map $\varphi_x : C(\sigma(A), \mathbb{R}) \to \mathbb{R}$ as follows,

$$\varphi_x(f) := (f(A)x \mid x).$$

Clearly φ_x is linear, as for any $f, g \in C(\sigma(A), \mathbb{R})$ and any $\alpha \in \mathbb{C}$ we have,

$$\begin{aligned}
\varphi_x(f + \alpha g) &= ((f + \alpha g)(A)x \mid x) \\
&= ((f(A) + \alpha g(A))(x) \mid x) \\
&= (f(A)x + \alpha g(A)x \mid x) \\
&= (f(A)x \mid x) + \alpha(g(A)x \mid x) = \varphi_x(f) + \alpha\varphi_x(g).
\end{aligned}$$

Now, suppose $f \in C(\sigma(A), \mathbb{R})$ and suppose $f(x) \geq 0$ for all $x \in \sigma(A)$, i.e.f is non-negative. By 4.33 (*iii*), it follows that $f(A)$ is a non-negative operator, i.e $f(A) \geq 0$. Thus, $\varphi_x(f) = (f(A)x \mid x) \geq 0$, and so φ_x satisfies the criteria of 4.34. It follows that for each $x \in H$, there is a unique measure μ_x on the Borel $\sigma-$algebra in $\sigma(A)$ such that

$$(f(A)x \mid x) = \int_{\sigma(A)} f(\lambda) d\mu_x(\lambda). \tag{4.5}$$

Furthermore, defining $\mathbf{1} : \sigma(A) \to \mathbb{R}$, by $\mathbf{1}(x) = 1$ clearly gives $\mathbf{1}(A) = \mathrm{id}_H$ and so we get the following equality,

$$\begin{aligned}
\varphi_x(\mathbf{1}) = (\mathbf{1}x \mid x) &= (x \mid x) \\
&= \int_{\sigma(A)} \mathbf{1}(\lambda)\, d\mu_x(\lambda) \\
&= \int_{\sigma(A)} d\mu_x \\
&= \mu_x(\sigma(A)).
\end{aligned} \tag{4.6}$$

Thus, we have that for each $x \in H$, $\|x\|^2 = \mu_x(\sigma(A))$.

Definition 4.36. Let $f : \sigma(A) \to \mathbb{C}$ be bounded and measurable. We let $\zeta_f : H \to \mathbb{C}$ be the map given as follows,

$$\zeta_f(x) = \int_{\sigma(A)} f(\lambda) d\mu_x(\lambda),$$

where μ_x is the measure from example 4.35.

If $f \in C(\sigma(A), \mathbb{R})$ then $\zeta_f(x) = (f(A)x \mid x)$ by 4.35. Thus, by A.41, ζ_f is a bounded quadratic form on H. We want to extend this result to all bounded, measurable, complex-valued functions. For this we will need the following lemma.

Lemma 4.37. *Let X be a compact metric space and suppose*

$$\mathcal{F} \subseteq \{f \in \mathbb{C}^X \mid f \textit{ is bounded and Borel measurable}\}.$$

Furthermore, suppose $\mathcal{F}$ has the following properties.

(i) $\mathcal{F}$ is a complex vector space containing $C(X, \mathbb{R})$.

(ii) For any $f \in \mathbb{C}^X$, if there is a uniformly bounded sequence $(f_n)_{n\in\mathbb{N}} \in \mathcal{F}^{\mathbb{N}}$ such that for all $x \in X, \lim_{n\to\infty} f_n(x) = f(x)$, then $f \in \mathcal{F}$.

Then, $\mathcal{F}$ consists of all bounded, Borel-measurable functions on X.

Proof: We prove this in a number of stages. First, let L_0 be the collection of all Borel-measurable subsets E of X such that 1_E is a uniformly bounded pointwise limit of a sequence of continuous functions, where 1_E denotes the characteristic function on E.
We show this is an algebra on X. First, X is trivially open, hence in the Borel $\sigma-$algebra on $\mathbb{C}$. Since $1_X(x) = 1$ for all $x \in X$, the constant 1 sequence $f_n : X \to \mathbb{C}$, $f_n(x) = 1$ satisfies the criteria.

Suppose A and B are both in L_0, so they are both measurable and there are sequences of uniformly bounded continuous functions from X to $\mathbb{C}$, $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ respectively, such that for all $x \in X$,

$$\lim_{n\to\infty} a_n(x) = 1_A(x), \text{ and } \lim_{n\to\infty} b_n(x) = 1_B(x).$$

Define for each natural number n a function $g_n : X \to \mathbb{C}$ by $g_n(x) = \max\{a_n(x), b_n(x)\}$. Since a_n and b_n are bounded for all $n \in \mathbb{N}$, so too is g_n. Note that we can rewrite g_n as follows,

$$g_n(x) = \max\{a_n(x), b_n(x)\} = \frac{1}{2}\left[a_n(x) + b_n(x)\right] + |a_n(x) - b_n(x)|,$$

and since a_n and b_n are continuous, it follows g_n is continuous also. Lastly, we also have $\lim_{n\to\infty} g_n(x) = 1_{A\cup B}(x)$, showing $A \cup B \in L_0$. Next, define a function $h_n : X \to C$, $h_n(x) = \max\{a_n(x) - b_n(x), 0\}$. Then $(h_n)_{n\to\infty}$ is a uniformly bounded sequence of continuous functions such that $\lim_{n\to\infty} h_n(x) = 1_{A\setminus B}(x)$ for all $x \in X$. It follows $A \setminus B \in L_0$, hence L_0 is an algebra as claimed.

We now show that L_0 contains all the open sets in X. Suppose $O \subseteq X$ is open. For each $n \in \mathbb{N}$, let $F_n = \{x \in O \mid d(x, X \setminus O) \geq 2^{-n}\}$. Each F_n is closed and $O = \bigcup_{n\in\mathbb{N}} F_n$. Moreover, $X \setminus O$ is closed and disjoint from each F_n. By Uhrysohn's lemma, for each $n \in \mathbb{N}$ there exists $f_n \in C(X, \mathbb{R})$ such that $f_n(x) = 1$ for all $x \in F_n$, and $f_n(x) = 0$ for all $x \in X \setminus O$. We also have that $(f_n)_{n\in\mathbb{N}}$ is uniformly bounded and $f_n \to 1_O$ pointwise. Therefore L_0 contains all the open subsets of X.

Next, let L_1 be the collection of all measurable subsets E of X, such that 1_E belongs to $\mathcal{F}$, that is,

$$L_1 = \{E \subseteq X \mid E \text{ is measurable and } 1_E \in \mathcal{F}\}.$$

We show that L_1 is a monotone class. Let $(A_n)_{n\in\mathbb{N}} \in {L_1}^{\mathbb{N}}$ and suppose it is increasing, i.e. $A_1 \subseteq A_2 \subseteq \dots$. Then $A := \bigcup_{n=1}^{\infty} A_n$ is measurable since $\sigma-$algebras are closed under countable unions. Moreover, 1_A and 1_B are the pointwise limits of the characteristic functions of A_n and B_n respectively. Since pointwise limits of measurable functions are measurable, it follows $1_A, 1_B \in L_1$.

Next, let $E \in L_0$. Then E is measurable and 1_E is a uniformly bounded limit of a sequence of continuous functions. Since $\mathcal{F}$ is closed under pointwise limits of uniformly bounded sequences, it follows that $1_E \in \mathcal{F}$ and so $E \in L_1$, i.e. $L_0 \subseteq L_1$. Thus, by the monotone class lemma A.32, it follows that L_1 contains the σ-algebra generated by L_0. Since L_0 contains all the open sets in X, this in turn means that L_1 contains all the Borel sets in X.

Lastly, suppose $f \in \mathbb{C}^X$ is bounded and measurable. Let $O \subseteq \mathbb{C}$ be a Borel set. Then $f^{-1}(O)$ is a Borel set in X. Thus, $f^{-1}(O) \in L_1$ and so $1_{f^{-1}(O)} \in \mathcal{F}$. It follows that $f \in \mathcal{F}$, and so $\mathcal{F}$ consits of all bounded, Borel-measurable functions from X to $\mathbb{C}$.

∎

Theorem 4.38. *If $f : \sigma(A) \to \mathbb{C}$ is bounded and measurable, then ζ_f is a bounded quadratic form.*

Proof: Let $\mathcal{F}$ be the collection of all bounded, measurable functions $f : \sigma(A) \to \mathbb{C}$ for which ζ_f is a quadratic form. Then it is easy to see that $\mathcal{F}$ is a vector space containing $C(\sigma(A), \mathbb{R})$ by 4.35. Suppose $(f_n)_{n\in\mathbb{N}} \in \mathcal{F}^{\mathbb{N}}$ is a sequence of uniformly bounded functions in $\mathcal{F}$ converging pointwise to a function $f \in \mathbb{C}^X$. Then f is bounded and measurable also. Furthermore, since each f_n is measurable and uniformly dominated by a constant, Lebesgue's dominated convergence theorem A.34, gives $\lim_{n\to\infty} \int_{\sigma(A)} f_n(\lambda) d\mu_x(\lambda) = \int_{\sigma(A)} f(\lambda) d\mu_x(\lambda)$, and so ζ_f is a quadratic form. It follows by 4.37 that $\mathcal{F}$ is the space of all bounded Borel-measurable functions.

Moreover, using 4.6 we get the following,

$$\begin{aligned} |Q_f(x)| &= \left| \int_{\sigma(A)} f(\lambda) d\mu_x(\lambda) \right| \\ &\leq \sup_{\lambda\in\sigma(A)} |f(\lambda)| \mu_x(\sigma(A)) \\ &= \sup_{\lambda\in\sigma(A)|} |f(\lambda)| \|x\|^2. \end{aligned}$$

Hence, ζ_f is a bounded quadratic form for every $f \in \mathcal{F}$. ■

Corollary 4.38.1. *Let $f : \sigma(A) \to \mathbb{C}$ be bounded and measurable, where $A \in B(H)$ is a self-adjoint operator. Then there exists a unique operator $f(A) \in B(H)$ such that for all $x \in H$ we have the following,*

$$(f(A)x \mid x) = Q_f(x) = \int_{\sigma(A)} f \, d\mu_x.$$

Furthermore, if $f(\sigma(A)) \subseteq \mathbb{R}$, then $f(A)$ is self-adjoint.

Proof: Combine theorem 4.38 with A.42. ■

Proposition 4.39. *Let $f, g \in \mathbb{C}^{\sigma(A)}$ be bounded and measurable functions. Then the functional calculus on A has the following property,*

$$(fg)(A) = f(A)g(A).$$

Proof: Let $\mathcal{F}_1$ denote the space of all bounded measurable functions f such that $(fg)(A) = f(A)g(A)$ for all $g \in C(\sigma(A), \mathbb{R})$. It is clear that $\mathcal{F}_1$ is a vector space containing $C(\sigma(A), \mathbb{R})$. As noted in 4.38 Lebesgue's dominated convergence theorem ensures A.34 the map $f \mapsto \zeta_f(x)$ for $x \in H$ is continuous under uniformly bounded, pointwise convergence. Let L_f be the associated sesquilinear form to ζ_f, and suppose $(f_n)_{n\in\mathbb{N}}$ is a sequence of uniformly bounded functions that converge pointwise to f, each with associated sesquilinear form L_{f_n}. By the polarisation identity 2.6, we have the following,

$$\begin{aligned} L_{f_n}(x,y) &= \frac{1}{2}\left[\zeta_{f_n}(x+y) - \zeta_{f_n}(x) - \zeta_{f_n}(y)\right] - \frac{i}{2}\left[\zeta_{f_n}(x+iy) - \zeta_{f_n}(x) - \zeta_{f_n}(iy)\right] \\ &\longrightarrow \frac{1}{2}\left[\zeta_f(x+y) - \zeta_f(x) - \zeta_f(y)\right] - \frac{i}{2}\left[\zeta_f(x+iy) - \zeta_f(x) - \zeta_f(iy)\right] \\ &= L_f(x,y) \end{aligned}$$

Thus, the map $f \mapsto L_f(x,y)$ is also continuous under uniformly bounded pointwise limits. By A.40, $f \in \mathcal{F}_1$ if and only if for every $g \in C(\sigma(A),\mathbb{R})$ and every $x \in H$, we have

$$((fg)(A)x \mid x) = (f(A)g(A)x \mid x)$$

which is equivalent to $\zeta_{fg}(x) = L_f(g(A)x, x)$. By the continuity of $f \mapsto L_f(x,y)$ and $f \mapsto \zeta_f(x)$ with respect to uniformly bounded pointwise limits, it follows that $\mathcal{F}_1$ is also continuous in this sense. Thus, using the result of 4.37, it follows that $\mathcal{F}_1$ consists of all bounded, Borel-measurable functions.

Next, let $\mathcal{F}_2$ be the space of all bounded, Borel-measurable functions f such that $(fg)(A) = f(A)g(A)$, for all bounded and Borel-measurable functions $g \in \mathbb{C}^{\sigma(A)}$. The above argument shows that $C(\sigma(A),\mathbb{R}) \subseteq \mathcal{F}_1$. Repeating the same argument as above and using 4.37 once more, we conclude that $\mathcal{F}_2$ consists of all bounded, Borel-measurable functions, proving the claim.

■

We are now ready to prove the main result of this thesis, the spectral theorem for self-adjoint operators.

Theorem 4.40. *Suppose A is a self-adjoint operator on a Hilbert space H. For all Borel-measurable sets $E \subseteq \sigma(A)$, define an operator $\mu^A(E)$ on H by*

$$\mu^A(E) := 1_E(A),$$

where $1_E(A)$ is the operator described by 4.38.1. Then μ^A is a projection-valued measure on $\sigma(A)$ and satisfies the following condition,

$$\int_{\sigma(A)} \lambda \, d\mu^A(\lambda) = A.$$

Proof: For a measurable set $E \subseteq \sigma(A)$, 1_E is real-valued and clearly satisifes $1_E \cdot 1_E = 1_E$. From the properties of the functional caclulus on A 4.33, we therefore have that $1_E(A)$ is self-adjoint and $1_E(A) = 1_E(A)^2$. Thus, $\mu^A(E)$ is an orthogonal projection

for any measurable set $E \subseteq \sigma(A)$.
Next, note that $1_{\emptyset}(x) = 0$ for all $x \in \sigma(A)$, and so $\mu^A(\emptyset) = 1_{\emptyset}(A) = \mathbf{0}$. We also clearly have $1_{\sigma(A)}(x) = 1 = x^0$ for all $x \in \sigma(A)$, and so $\mu^A(\sigma(A)) = 1_{\sigma(A)}(A) = A^0 = \mathrm{id}_{\sigma(A)}$.
If E_1 and E_2 are both Borel sets, then $1_{E_1 \cup E_2}(x) = 1_{E_1}(x) \cdot 1_{E_2}(x)$ for all $x \in \sigma(A)$, hence $\mu^A(E_1 \cup E_2) = \mu^A(E_1)\mu^A(E_2)$.
Suppose $(E_n)_{n \in \mathbb{N}}$ is a sequence of disjoint Borel sets in $\sigma(A)$. We must show that for all $x \in H$, the following holds,

$$\mu^A\left(\bigcup_{n=1}^{\infty} E_n\right) x = \sum_{n=1}^{\infty} \mu^A(E_n)x.$$

First, note that if $m \neq n$, then $\mu^A(E_n)\mu^A(E_m) = \mu^A(\emptyset) = \mathbf{0}$. Since $\mu^A(E_n)$ and $\mu^A(E_m)$ are projections, it follows that $\mathrm{Ran}(\mu^A(E_n))^{\perp} = \mathrm{Ran}(\mu^A(E_m))$. Next let $\mathcal{F} = \bigcap \left\{ S \subseteq H \mid S \text{ is a closed subspace of } H \text{ and } \mathrm{Ran}(\mu^A(E_n)) \subseteq S \text{ for all } n \in \mathbb{N} \right\}$. $\mathcal{F}$ is a closed subspace of H and so let P be the projection onto $\mathcal{F}$. It then follows that for all $x \in H$,

$$Px = \lim_{n \to \infty} \sum_{k=1}^{n} \mu^A(E_k)x.$$

If $E := \bigcup_{n=1}^{\infty} E_n$ then the sequence of functions $f_n = \sum_{k=1}^{n} 1_{Ek}$ is uniformly bounded by 1 and converges pointwise to 1_E. Using Lebesgue's dominated convergence theorem A.34, we therefore, have that

$$\lim_{n \to \infty} \int_{\sigma(A)} f_n d\mu_x = \int_{\sigma(A)} 1_E \, d\mu_x.$$

Using 4.38.1, for all $x \in H$ we have

$$\lim_{n \to \infty} \left(\sum_{k=1}^{n} 1_{E_k}(A)x \mid x \right) = (1_E(A)x \mid x).$$

It follows that $1_E(A)x = Px$ for all $x \in H$, i.e. $\mu^A\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu^A(E_n)$. Thus, μ^A is a projection-valued measure.
Next, suppose $g \in \mathbb{C}^{\sigma(A)}$ is a simple function, so $g = \sum_{k=1}^{n} a_k 1_{E_k}$ for a disjoint collection of Borel sets $(E_k)_{k=1}^{n}$. We then have the following set of equations.

$$\begin{aligned}
\int_{\sigma(A)} g(\lambda)\, d\mu^A(\lambda) &= \int_{\sigma(A)} \sum_{k=1}^{n} a_k 1_{E_k}\, d\mu^A(\lambda) \\
&= \sum_{k=1}^{n} a_k \int_{\sigma(A)} 1_{E_k}\, d\mu^A(\lambda) \\
&= \sum_{k=1}^{n} a_k\, \mu^A(E_k) \\
&= \sum_{k=1}^{n} a_k 1_{E_k}(A) = g(A).
\end{aligned}$$

Now suppose $f \in \mathbb{C}^{\sigma(A)}$ is bounded and measurable. Then f is the uniform limit of a sequence of simple functions $(f_n)_{n\in\mathbb{N}}$, by theorem A.33. This, combined once more with Lebesgue's dominated convergence theorem A.34 therefore gives,

$$\begin{aligned}
\int_{\sigma(A)} f(\lambda) d\mu^A(\lambda) &= \int_{\sigma(A)} \lim_{n\to\infty} f_n(\lambda)\, d\mu^A(\lambda) \\
&= \lim_{n\to\infty} \int_{\sigma(A)} f_n(\lambda)\, d\mu^A(\lambda) \\
&= \lim_{n\to\infty} f_n(A) = f(A).
\end{aligned}$$

Since the inclusion map $f(\lambda) = \lambda$ is trivially a bounded and measurable function on $\sigma(A)$, and $f(A) = A$, the previous set of computations gives

$$\int_{\sigma(A)} \lambda\, d\mu^A(\lambda) = A.$$

■

Remark 4.41. The spectral measure μ^A in theorem 4.40 is unique. This is a consequence of the measure-theoretic Riesz representation theorem 4.34 and equation 4.5.

5 Conclusion

The spectral theorem from the final section is the beginning of a vast area of research in mathematics known as operator theory which studies objects known as C^* algebras. In particular, there are formulations of the spectral theorem proven entirely within the context of C^* algebras, see section 2.5 in [13]. If I had more time to explore further topics this is undoubtedly the route I would have gone down. Besides this, there is another formulation of the spectral theorem I would have enjoyed proving involving a concept known as direct integrals, see theorem 7.19 in [9]. Other topics of further study I would have considered include Fredholm theory and applications to quantum mechanics.

The theory of Hilbert space operators is indispensable in many areas of mathematics. Given that I will be beginning postgraduate studies this September in the area of Harmonic Analysis, I am grateful that I was given the opportunity to do this project and learn everything I learned.
Furthermore, I was lucky enough to have been able to study Functional Analysis as well as Measure and Integration theory before starting with my project. Without these modules, it would have been impossible to complete.

As a final note, I'd like to extend sincere gratitude to my project supervisor Prof. Martin Mathieu whose guidance this year was invaluable.

A Appendix

This appendix serves to provide key theorems and results that are of fundamental importance throughout the theory of Hilbert space operators, but whose proofs and development would make this thesis less focused and too expansive, and as a result, we include them here.

A.1 Concepts in Set Theory and Topological Spaces.

The following section uses [12] and [18] as primary guiding sources.

Definition A.1. Let $(X, \leq)$ be a partially ordered set.

- A *chain* in X is a subset that is totally ordered.
- An *upper bound* for a subset $Y \subseteq X$ is an element $x \in X$ such that $y \leq x$ for all $y \in Y$.
- X is *inductively ordered* if every chain in X has an upper bound.
- A *maximal element* for a subset $Y \subseteq X$ is an element $y \in Y$ such that if $y \leq x$ for some $x \in Y$, then $x = y$.

Axiom A.2 (Zorn's Lemma)**.** *Every non-empty inductively ordered set contains a maximal element.*

Definition A.3. Let X be a topological space. We say that X is *compact* if each of its open covers has a finite subcover. That is, X is compact if for every collection $\mathcal{C}$ of open subsets of X such that

$$X = \bigcup_{A \in C} A,$$

there exists a finite subset $F \subseteq \mathcal{C}$ such that

$$X = \bigcup_{A \in F} A.$$

Definition A.4. Let (X, d) be a metric space. We say that a subset A is *precompact* if $\overline{A}$ is compact.

Definition A.5. A metric space (X, d) is *complete* if every Cauchy sequence in X converges to a point in X.

Definition A.6. We say a subset Y of a metric space (X, d) is *dense* in X, if the closure of Y is X, that is, $\overline{Y} = X$.

Definition A.7. Let (X, d) be a metric space. We say that X is *separable*, if it has a subset $Y \subseteq X$ such that Y is dense in X.

Definition A.8. Let Y be a subset of the metric space (X, d). We say that

(i) Y is *rare* (*nowhere dense*) if $\overline{Y}$ has empty interior.

(ii) Y is *meagre* (*of first category*) if $Y = \bigcup_{n \in \mathbb{N}} Y_n$ for a countable family of rare subsets Y_n.

(iii) Y is *non-meagre*(*of second category*) if Y is not meagre.

Theorem A.9. *Let (X, d) be a metric space. If $Y \subseteq X$ is a meagre subset of X, then Y^c is dense.*

Theorem A.10 (Baire Category Theorem)**.** *Every complete metric space is of second category.*

Definition A.11. Let (X, τ) be a topological space. We say X is *normal* if given any disjoint, closed sets E and F, there are neighbourhoods U of E and V of F respectively, such that U and V are disjoint.

Theorem A.12 (Urysohn's Lemma)**.** *Let (X, τ) be a topological space. Then X is normal if and only if for any two non-empty, closed, disjoint subsets E and F of X, there exists a continuous map $f : X \to [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.*

A.2 Concepts in Functional Analysis

In this section the main theorems and concepts from functional analysis are provided. See [1], [11] and [5] for proofs and general theory.

Definition A.13. Let E be a complex vector space. A function $\|\cdot\| : E \to [0, \infty) \subseteq \mathbb{R}$ that satisfies the following three axioms for all $x, y \in E$ and $\alpha \in \mathbb{C}$ is called a *norm* on E.

(i) $\|x\| = 0$ if and only if $x = 0$.

(ii) $\|\alpha x\| = |\alpha|\|x\|$.

(iii) $\|x + y\| \leq \|x\| + \|y\|$.

In this case we say E is a *normed vector space* or simply a *normed space* for short.

Given a normed space $(E, \|\cdot\|)$, the relation $d(x, y) := \|x - y\|$ defines a metric d on E. If every cauchy sequence in a normed space E converges to a point in E with respect to the induced metric d, then we say that E is a *Banach space.*

Definition A.14. Let E and F be Banach spaces. We say that a linear map $T : E \to F$ is *bounded* if there exists $M > 0$ such that $\|Tx\| \le M\|x\|$ for all $x \in E$.

We denote the set of all bounded linear maps from the Banach space E into the Banach space F by $B(E, F)$. Under the usual operations of addition and scalar multiplication by elements in $\mathbb{C}$, $B(E, F)$ is a vector space. In addition, we can equip it with the following norm. Let $T \in B(E, F)$, and let E_1 denote the unit ball of E, that is, $E_1 = \{x \in E \mid \|x\| \le 1\}$. We then define $\|T\|$ as the following,

$$\|T\| := \sup\{\|T(x)\| : x \in E_1\}.$$

It can be shown that this norm is equivalent to the following quantity,

$$\|T\| = \inf\{M \in \mathbb{R}^+ \mid \|Tx\| \le M\|x\|, \text{ for all } x \in E\}.$$

Theorem A.15. *Suppose $T : E \to F$ is a linear mapping from the Banach space E into the Banach space F. Then T is bounded if and only if it is continuous.*

Proof: If T is the zero operator then it is obviously continuous, so suppose T is not the zero operator, is bounded, and let $\varepsilon > 0$. Choose $\delta = \frac{\varepsilon}{\|T\|}$. Then for $x, y \in E$, we have

$$\|x - y\| < \delta \implies \|T\|\|x - y\| < \varepsilon.$$

Since $\|Tx - Ty\| = \|T(x - y)\| \le \|T\|\|x - y\| < \varepsilon$, it follows that T is in fact uniformly continuous, hence continuous.
In the reverse direction, suppose T is continuous. In particular, T is continuous at $0 \in E$ and hence there exists $\delta > 0$ such that for all $x \in E_\delta$, $Tx \in F_1$. Without loss of generality, suppose that $x \neq 0$. We then have the following,

$$\|Tx\| = \left\| \frac{\|x\|}{\delta} T\left(\frac{\delta}{\|x\|}x\right) \right\| = \frac{\|x\|}{\delta} \left\| T\left(\frac{\delta}{\|x\|}x\right) \right\| \le \frac{\|x\|}{\delta} \cdot 1 = \frac{1}{\delta}\|x\|.$$

Thus for all $x \in E$ we have $\|Tx\| \le \frac{1}{\delta}\|x\|$ and so T is bounded. ∎

Theorem A.16. *Let $T : \mathbb{C}^n \to \mathbb{C}^n$ be linear map, where $\mathbb{C}^n$ is equipped with the standard inner product. Then T is bounded.*

Theorem A.17. *Let E be a finite-dimensional normed space. Then E is complete.*

Definition A.18. Let E be a Banach space over a field $\mathbb{K}$. The dual space of E, denoted by E', is the set of all continuous linear operators from E into $\mathbb{K}$, i.e,

$$E' := \{f \in \mathbb{K}^E \mid f \text{ is continuous and linear }\}. \tag{A.1}$$

The elements of E' are called *bounded linear functionals.*

Definition A.19. A linear map $T : E \to F$ between normed spaces E and F is called an *isometry* if $\|Tx\| = \|x\|$ for all $x \in E$.

Theorem A.20 (Hahn–Banach)**.** *Let E_0 be a subspace of normed space E. For every $f_0 \in E_0'$ there exists $f \in E'$ such that $f|_{E_0} = f_0$ and $\|f\| = \|f_0\|$, where $f|_{E_0}$ denotes the restriction of f to E_0.*

Theorem A.21. *Suppose X is a normed space. Then X is a Banach space if and only if every absolutely convergent series converges.*

Theorem A.22 (Uniform Boundedness Principle)**.** *Let E be a Banach space and let F be a normed space. Let $\mathcal{T} \subseteq \mathcal{L}(E, F)$ and suppose* $\sup\{\|Tx\| \mid T \in \mathcal{T}\} < \infty$ *for all $x \in E$. Then* $\sup\{\|T\| \mid T \in \mathcal{T}\} < \infty$.

Theorem A.23 (Bounded Inverse Theorem)**.** *Let X and Y be Banach spaces, and suppose $T : X \to Y$ is bounded and linear. If T is bijective, then $T^{-1} : Y \to X$ is also bounded.*

Theorem A.24. *Suppose $T : X \to Y$ is a bounded linear operator, where X and Y are Banach spaces. Then T is not invertible if it is not bounded from below.*

Theorem A.25. *Let $A \in B(H)$ be a bounded linear operator on a Hilbert space H. If A is invertible, then there exists $\varepsilon > 0$ such that if B is a bounded linear operator and $\|A - B\| < \varepsilon$, then B is invertible.*

Theorem A.26 (Banach–Alaoglu)**.** *Let E be a normed space. Then the closed unit ball of E' is compact with respect to the weak-$*$ topology.*

Theorem A.27. *Let X be a separable normed space. Then every bounded sequence in X' has a subsequence that converges in X' with respect to the weak$-^*$ topology.*

Proof: See [15], chapter 8, section 3.1. Alternatively, see [3] chapter 8. . ∎

Definition A.28. Let E be a Banach space over $\mathbb{C}$. The bidual of E, denoted E'' is the set of all bounded linear maps from E' to $\mathbb{C}$, that is

$$E'' = \{f \in \mathbb{C}^{E'} \mid f \text{ is bounded and linear }\}. \tag{A.2}$$

For every normed space E there exists a canonical embedding $J : E \to E''$ given by $J(x)(f) := f(x)$, where $f \in E'$ and $x \in E$. Using the Hahn–Banach theorem A.20, it can be shown that this map is a linear isometry. It is, however, not always surjective. Spaces for which this is the case are given a special name.

Definition A.29. Let E be a normed space and let $J : E \to E''$ be the canonical embedding of E into its bidual. We say that E is *reflexive* if J is surjective.

It is not always the case however, that if a normed space is isometrically isomorphic to its bidual, that it is reflexive. The prototypical example of such a space was given by R.C James, and is aptly named the *James space*. A detailed descriton can be found in [5], chapter 2, section 4.

Let E be a normed space and let $J : E \to E''$ be the canonical embedding of E into its bidual. We call the initial topology on E with respect to all functionals $f \in E'$ the weak topology on E, i.e, it is the weakest topology on E making all $f : E \to \mathbb{C}$ continuous. The weak topology is denoted by $\sigma\left(E, E'\right)$.
In a similar vein, the initial topology on E' with respect to all functionals $Jx : E' \to \mathbb{C}$ is called the weak-$*$ topology on E'. It is the weakest topology on E' ensuring that every $Jx : E' \to \mathbb{C}$ is continuous, as x ranges through E. The weak-$*$ topology is denoted by $\sigma(E^*, E)$.

Theorem A.30 (Continuity of the inner product). *Let $(E, (\cdot \mid \cdot))$ be an inner product space, and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in E converging to a point $x \in E$. Then for all $y \in E$, $\lim_{n\to\infty}(x_n \mid y) = (x \mid y)$.*

Proof: For a fixed $y \in E$, define a map $f_y : E \to \mathbb{C}$ by $f_y(x) = (x \mid y)$. Cleary f_y is linear and by the Cauchy–Schwarz inequality 2.8, we have that $|f_y(x)| \leq \|y\|\|x\|$ and so f_y is bounded. By theorem A.15 it follows that f_y is continuous, hence $\lim_{n\to\infty} f_y(x_n) = \lim_{n\to\infty}(x_n \mid y) = (x \mid y) = f_y(x)$.

■

A.3 Results from Measure Theory

Measure theoretic concepts are used in the formulation and proof of the spectral theorem for self-adjoint operators, and as such, here we state a few key concepts used throughout section 4.3. More in-depth formulations included with proofs can be found in [2]. In particular, a proof of the monotone class lemma A.32 is included in section 5.A, and a proof of A.33 can be found in section 2.E

Definition A.31. Let X be a set and let M be a collection of subsets of X, and suppose $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$ are sequences of sets within M such that $(A_n)_{n\in\mathbb{N}}$ is increasing and $(B_n)_{n\in\mathbb{N}}$ is decreasing. Then M is a *monotone class* on X, if

$$\bigcup_{n=1}^{\infty} A_n \in M \text{ and } \bigcap_{n=1}^{\infty} B_n \in M.$$

Lemma A.32 (Monotone Class Lemma). *Suppose M is a monotone class on a set X, and suppose A is an algebra contained in M. Then M contains the $\sigma-$algebra generated by A.*

Theorem A.33 (Approximation by Simple Functions)**.** *Suppose* (X, Σ) *is a measurable space and* $f : X \to [-\infty, \infty]$ *is* $\Sigma-$*measurable. Then there exists a sequence of functions* $(f_n)_{n\in\mathbb{N}} \in \left(\mathbb{R}^X\right)^{\mathbb{N}}$ *such that*

(i) Each f_n *is a simple* $\Sigma-$*measurable function.*

(ii) For each $n \in \mathbb{N}$ *and* $x \in X$*, we have* $|f_k(x)| \leq |f_{k+1}(x)| \leq |f(x)|$.

(iii) For each $x \in X : \lim_{n\to\infty} f_n(x) = f(x)$

(iv) If f *is bounded, the sequence* $(f_n)_{n\to\infty}$ *converges to* f *uniformly.*

The following core result in measure theory gives conditions under which we may swap an integral with limits. A proof can be found in section $3.B$ of [2].

Theorem A.34 (Lebesgue's Dominated Convergence Theorem)**.** *Suppose* (X, Σ, μ) *is a measure space and* $f : X \to [-\infty, \infty]$ *is* $\Sigma-$*measurable. Suppose furthermore that* $(f_n)_{n\in\mathbb{N}}$ *is a sequence of* $\Sigma-$*measurable functions from* X *to* $[-\infty, \infty]$ *such that*

$$\lim_{n\to\infty} f_n(x) = f(x)$$

for almost every $x \in X$*. If there exists a* $\Sigma-$*measurable function* $g : X \to [-\infty, \infty]$ *such that*

$$\int g \, d\mu < \infty \text{ and } |f_n(x)| \leq g(x)$$

for almost every $x \in X$ *and every* $n \in \mathbb{N}$*, we then have that*

$$\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu.$$

A.4 Quadratic and Sesquilinear Forms.

The basic theory of quadratic and sesquilinear forms is found throughout functional analysis, and as such we require it to develop the theory of operators on Hilbert space. A guiding source for the material here can be found in section $A.4.4$ in [9].

Definition A.35. Let H be a complex Hilbert space. A map $\phi : H \times H \to \mathbb{C}$ is called a *sesquilinear form on* H if it is linear with respect to the first argument and antilinear with respect to the second, that is, for all $x, y, z \in H$ and $\alpha, \beta \in \mathbb{C}$ we have

(i) **Linear in the first argument:**

$$\phi(\alpha x + \beta y, z) = \alpha\phi(x, z) + \beta\phi(y, z).$$

(ii) **Antilinear in the second argument:**

$$\phi(x, \alpha y + \beta z) = \overline{\alpha}\phi(x, y) + \overline{\beta}\phi(x, z).$$

It is easily recognised that the familiar inner product is a sesquilinear form.

Definition A.36. A sesquilinear form ϕ on a Hilbert space H is *bounded* if there exists $M > 0$ such that for all $x, y \in H$, we have

$$|\phi(x, y)| \leq M\|x\|\|y\|.$$

Proposition A.37 (Polarisation Identity)**.** *If ϕ is a sesquilinear form on a Hilbert space H, then for all $x, y \in H$ we have the following,*

$$\begin{aligned}\phi(x, y) = &\frac{1}{2}\left[\phi(x+y, x+y) - \phi(x, x) - \phi(y, y)\right] \\ &- \frac{i}{2}\left[\phi(x+iy, x+iy) - \phi(x, x) - \phi(iy, iy)\right].\end{aligned}$$

Definition A.38. Let H be a complex Hilbert space. A functional $Q : H \to \mathbb{C}$ is a *quadratic form on H* if it satisfies two conditions, namely, for all $x \in H$ and $\alpha \in \mathbb{C}$, $Q(\alpha x) = |\alpha|^2 Q(x)$, and the following map $L : H \times H \to \mathbb{C}$ given by

$$\begin{aligned}L(x, y) = &\frac{1}{2}\left[Q(x+y) - Q(x) - Q(y)\right] \\ &- \frac{i}{2}\left[Q(x+iy) - Q(x) - Q(iy)\right]\end{aligned}$$

is a sesquilinear form. Furthermore, we say Q is *bounded* if there is $M > 0$ such that for all $x \in H$,

$$|Q(x)| \leq C\|x\|^2.$$

Proposition A.39. *Let Q be a quadratic form on a Hilbert space H with associated sesquilinear form L. We then have the following.*

(i) For all $x \in H$, $Q(x) = L(x, x)$.

(ii) If Q is bounded then L is bounded.

(iii) If $Q(x) \in \mathbb{R}$ for all $x \in H$, then L is conjugate symmetric, i.e.

$$L(x, y) = \overline{L(y, x)}, \text{ for all } x \in H.$$

Theorem A.40. *Suppose L_1 and L_2 are sesquilinear forms on a Hilbert space H. If $L_1(x,x) = L_2(x,x)$ for all $x \in H$, then $L_1(x,y) = L_2(x,y)$ for all $x,y \in H$. Consequently, if A and B are operators on H such that $(Ax \mid x) = (Bx \mid x)$ for all $x \in H$, then $A = B$.*

Example A.41. Let $A \in B(H)$ be an operator on a Hilbert space H. Then the map $Q_A : H \to \mathbb{C}$ given by

$$Q_A(x) = (Ax \mid x)$$

is a bounded quadratic form with associated sesquilinear form L_A given by

$$L_A(x,y) = (Ax \mid y).$$

Theorem A.42. *If Q is a bounded quadratic form on a Hilbert space H, then there is a unique operator $A \in B(H)$ such that $Q(x) = (Ax \mid x)$ for all $x \in H$. Furthermore, if $Q(x) \in \mathbb{R}$ for all $x \in H$, then A is self-adjoint.*

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